

# PROPERLY EMBEDDED MINIMAL ANNULI IN $\mathbb{S}^2 \times \mathbb{R}$

L. HAUSWIRTH, M. KILIAN, AND M. U. SCHMIDT

ABSTRACT. We prove that properly embedded minimal annuli in  $\mathbb{S}^2 \times \mathbb{R}$  are foliated by circles. We use the compactness of the moduli of embedded annuli with flux bounded away from zero, and the property that the third component of the flux attains its maximum on the flat cylinder. We present an integrable systems approach and the Whitham deformation to deform minimal annuli, increasing the flux on non-flat embedded annuli while preserving embeddedness.

## CONTENTS

1. Introduction	1
2. Proof of the Main Theorem	4
3. Minimal annuli and the sinh-Gordon equation	14
4. Curvature estimate of Meeks and Rosenberg	17
5. Finite type theory of the sinh-Gordon equation	19
6. Construction of minimal annuli via potentials	21
7. Isospectral group and spectral data	22
8. Isospectral group and embedded annuli	26
9. Deformation of finite type annuli	28
9.1. Flow without common roots of $a$ and $b$	30
9.2. Smooth parametrization of spectral data and opening double point	31
9.3. Flow with common roots of $a$ and $b$	33
10. Isolated property of the Abresch family	37
11. Bubbletons on Abresch family are not embedded	39
Appendix A. Sym point	41
Appendix B. Bubbletons	42
Appendix C. Abresch annuli	43
C.1. Spectral data	43
C.2. Whitham deformation of spectral genus 0, 1 and 2	44
References	45

## 1. INTRODUCTION

In this paper we discuss the geometry of properly embedded minimal annuli in  $\mathbb{S}^2 \times \mathbb{R}$ . There is a two-parameter family of embedded minimal annuli foliated by horizontal constant curvature curve of  $\mathbb{S}^2$ . The simplest non compact examples are the totally geodesic  $\Gamma \times \mathbb{R}$ , where  $\Gamma$  is a simple closed geodesic on  $\mathbb{S}^2$ . There exists a one-parameter family of periodic properly

embedded annuli which are small graphs over  $\Gamma \times \mathbb{R}$ . These examples were described analytically by Pedrosa and Ritore [20] and they called them unduloids. They appear in the isoperimetric profile of  $\mathbb{S}^2 \times \mathbb{S}^1$ . These examples are rotational surfaces around vertical geodesics. A one-parameter helicoidal family, obtained by rotating a great circle on  $\mathbb{S}^2$  at a constant rate in the third coordinate about an axis passing through a pair of antipodal points on the rotated great circle was constructed by Rosenberg [23].

A two-parameter family of deformations of previous examples was constructed by the first-named author [7], and by Meeks and Rosenberg (see [18], section 2) using variational arguments. This involves solving a Plateau problem with boundaries given by two geodesics  $\Gamma_1$  and  $\Gamma_2$  in parallel sections  $\mathbb{S}^2 \times \{t_1\}$  and  $\mathbb{S}^2 \times \{t_2\}$ . The stable annulus bounded by these geodesics is foliated by horizontal constant curvature curves (see theorem 3.3). Schwarz symmetry along boundary geodesics then gives a complete and properly embedded example. The two parameters (up to isometries of  $\mathbb{S}^2 \times \mathbb{R}$ ) of such compact annuli are the distance between the two sections including the boundary, and the position of one geodesic in  $\mathbb{S}^2 \times \{t\}$ , keeping the other fixed. They are periodic in the third direction, foliated by constant curvature curves of  $\mathbb{S}^2$  and have a vertical plane of symmetry. These examples are very similar to the minimal Riemann's staircase in  $\mathbb{R}^3$ . Constant mean curvature (CMC) tori in  $\mathbb{R}^3$  give rise to minimal annuli in  $\mathbb{S}^2 \times \mathbb{R}$  under certain conditions as follows. The Gauss map of a CMC torus is a harmonic map  $G : \mathbb{T}^2 \rightarrow \mathbb{S}^2$ . Its holomorphic quadratic differential is

$$Q = \langle G_z, G_z \rangle (dz)^2$$

where  $z = x + iy$  is a global holomorphic coordinate on the torus. Since  $\langle G_z, G_z \rangle : \mathbb{T}^2 \rightarrow \mathbb{C}$  is holomorphic, it is a non-zero constant  $\langle G_z, G_z \rangle \equiv c \in \mathbb{C}^*$ .

After a linear change of coordinate we can assume  $c = \pm 1/4$ . Then the map  $X : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$ ,

$$X(z) = (G(z), \operatorname{Re}(-2i\sqrt{c}z))$$

is conformal and harmonic, and thus locally a minimal surface in  $\mathbb{S}^2 \times \mathbb{R}$  (possibly branched). We can choose the sign of  $c \in \mathbb{R}$  in such a way that the large curvature line on the (CMC) torus corresponds to a horizontal curve in  $\mathbb{S}^2 \times \{t\}$ . In this case  $X(z)$  is an immersion. If the Gauss map  $G$  is periodic along this horizontal curve then we have a minimal annulus.

The flat CMC cylinder in  $\mathbb{R}^3$  yields a flat minimal annulus in  $\mathbb{S}^2 \times \mathbb{R}$ , and the Gauss maps of Delaunay surfaces ( $G$  is periodic as defined on a torus) yield the unduloid, and the CMC nodoids yield the helicoid in  $\mathbb{S}^2 \times \mathbb{R}$  under this correspondence by the harmonic map  $G$ .

Supplementing the rotational examples, there is a two-parameter family of harmonic maps studied by Abresch [1]. To describe the equations of Wente tori in  $\mathbb{R}^3$ , Abresch studies conformal CMC  $H = 1/2$  immersion with large line or small line of curvature included in a plane. He studies constant mean curvature surfaces parameterized by  $\mathbb{R}^2$ , with the coordinate axes  $x$  and  $y$  yielding the principal lines of curvature, and solves closing conditions for the surface to obtain CMC tori. On these lines, the Gauss map takes values in a constant curvature curve of  $\mathbb{S}^2$ .

In a conformal parametrization with  $|c| = 1/4$ , the metric of the minimal annulus in  $\mathbb{S}^2 \times \mathbb{R}$  is given by  $ds^2 = \cosh^2 \omega |dz|^2$  where  $\omega : \mathbb{C} \rightarrow \mathbb{R}$  is a solution of the sinh-Gordon equation

$$(1.1) \quad \Delta \omega + \sinh \omega \cosh \omega = 0,$$

where  $\Delta$  denotes the Laplacian of the flat metric  $|dz|^2$ . The metric of the corresponding CMC  $H = 1/2$  surface in  $\mathbb{R}^3$  is given by  $d\tilde{s}^2 = e^{2\omega} |dz|^2$ . Abresch classified all real solutions  $\omega : \mathbb{C} \rightarrow \mathbb{R}$  of the system

$$(I) \quad \begin{cases} \Delta \omega + \sinh \omega \cosh \omega = 0 \\ \sinh(\omega)(\omega_{xy}) - \cosh(\omega)(\omega_x)(\omega_y) = 0 \end{cases}$$

where the second equation is the condition that the small curvature lines are planar. Then he proves that this family contains CMC  $H = 1/2$  tori. These tori give doubly periodic harmonic maps  $G$  and so minimal immersions by considering  $X(z) = (G(z), x)$  (the map  $X(z) = (G(z), y)$  is branched). The horizontal curves of this family have non-constant curvature.

In a second part, Abresch studies solutions of the system

$$(II) \quad \begin{cases} \Delta\omega + \sinh\omega \cosh\omega = 0 \\ \cosh(\omega)(\omega_{xy}) - \sinh(\omega)(\omega_x)(\omega_y) = 0 \end{cases}$$

where the second equation is the condition that large lines of curvature are planar. These solutions  $\omega$  induces a CMC immersion of  $\mathbb{C}$  in  $\mathbb{R}^3$  and a doubly-periodic Gauss map  $G : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ . The second equation is the condition that the immersion  $X(z) = (G(z), y)$  has horizontal constant curvature curve and parameterizes the whole family of properly embedded annuli of  $\mathbb{S}^2 \times \mathbb{R}$ . For these reasons,

**Definition 1.1.** *An Abresch annulus of  $\mathbb{S}^2 \times \mathbb{R}$  is an embedded annulus foliated by horizontal constant curvature curves.*

It was conjectured by Meeks-Rosenberg [18] that any properly embedded genus zero minimal surface in  $\mathbb{S}^2 \times \mathbb{R}$  is an Abresch annulus. In this direction, Hoffmann and White [11] proved that if a properly embedded annulus of  $\mathbb{S}^2 \times \mathbb{R}$  contains a vertical geodesic, then this is a helicoid type example. The first author in [7] characterized Abresch annuli as the only annuli which are foliated by horizontal curvature curves. Our main result confirms this conjecture, and we prove the following

**Main Theorem.** *A properly embedded minimal annulus in  $\mathbb{S}^2 \times \mathbb{R}$  is an Abresch annulus.*

The proof combines methods from geometric analysis with techniques of integrable systems. The first ingredient is a linear area growth and curvature estimate of Meeks-Rosenberg. Due to theorem 7.1 in [17] (see Theorem 4.2 below), the curvature and thus the area growth of a properly embedded minimal annulus  $A$  in  $\mathbb{S}^2 \times \mathbb{R}$  are bounded by constants depending on the flux of the third coordinate  $h : A \rightarrow \mathbb{R}$  along horizontal sections. Properly immersed annuli in  $\mathbb{S}^2 \times \mathbb{R}$  are parabolic (see theorem 3.2) and the flux corresponds to the length of the period  $\tau$  of the corresponding solution of the sinh-Gordon equation (1.1)(see lemma 4.1). If the flux  $|\tau| \geq \epsilon_0$ , there is a constant  $C_1 > 0$  depending only on  $|\tau|$  such that

$$|K| \leq C_1(\epsilon_0)$$

We improve the linear area growth estimate of theorem 1.1 in [17] using parabolicity, and prove in lemma 4.3 that there is a constant  $C_2 > 0$  depending only on  $\epsilon_0$  such that for any  $t > 0$ ,

$$\text{Area}(A \cap [-t, t]) \leq C_2(\epsilon_0)t.$$

This estimate has two consequences. Firstly it implies that properly embedded annuli are of finite type. This means that the periodic immersion  $X : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  with period  $\tau \in \mathbb{C}^*$  can be described by algebraic data. Up to some finite dimensional and compact degree of freedom the immersion is determined by the so-called *spectral data*  $(a, b)$ . They consist of two polynomials of degree  $2g$  respectively  $g + 1$  for some  $g \in \mathbb{N}$ . The polynomial  $a(\lambda)$  encodes a hyperelliptic Riemann surface called *spectral curve*. The genus of the spectral curve is called *spectral genus*. The other polynomial  $b(\lambda)$  encodes the closing conditions  $X(z + \tau) = X(z)$  for some  $\tau \in \mathbb{C}^*$  depending on  $(a, b)$ . This correspondence is called the algebro-geometric correspondence.

Using this algebro-geometric correspondence, one can deform a minimal annulus by deforming the corresponding spectral data. Krichever introduced the Whitham deformations as a tool to deform spectral data. Starting with an embedded minimal annulus in  $\mathbb{S}^2 \times \mathbb{R}$ , the Whitham deformation allows us to deform the annulus preserving minimality, closing condition as well as

embeddedness. Applying this to the flat embedded minimal annulus allows us to flow through the path-connected component of embedded annuli. In this way we are able to construct the whole family of Abresch annuli via Whitham deformation theory (see Appendix C).

The second consequence of the curvature estimate is the compactness of the space of spectral data corresponding to properly embedded minimal annuli with periods  $\tau$  bounded away from zero. As a direct consequence each connected component of the space of such spectral data  $(a, b)$  contains a maximum of the period  $\tau$ , since connected components are always closed.

If the spectral genus is larger than zero, then we show in Lemma 9.1 that there always exists a Whitham deformation of  $(a, b)$  increasing the period  $\tau$ . The flat annulus is the only surface whose spectral curve has spectral genus zero. Therefore there is only one connected component of spectral data  $(a, b)$  corresponding to properly embedded minimal annuli.

Helicoidal and rotational unduloids are of spectral genus one, while the Riemann's type examples are of spectral genus two. This family of Abresch annuli is characterized by an additional symmetry in the corresponding spectral data  $(a, b)$ , and we will see that it is not possible to continuously break this symmetry while preserving a closing condition of the annulus. Therefore the Abresch annuli form the only connected component of the embedded flat minimal annuli.

The curvature and area growth estimate of Meeks-Rosenberg helps us to simplify the proof of the Main Theorem in the present paper. However such an estimate relating the flux and the curvature is not necessary to control the deformation. In [9] the present authors construct a Whitham deformation connecting the spectral data of an arbitrary Alexandrov embedded annulus in  $\mathbb{S}^3$  with the spectral data of the Clifford torus without using such an estimate. It is shown that no geometrical accident can appear along this deformation.

Each of the remaining sections contains a step of the proof. In the subsequent Section 2 we present the results of all remaining sections and explain how they fit together to a proof of the Main Theorem. Section 2 contains the proof of the Main Theorem and a summary of the remaining sections.

## 2. PROOF OF THE MAIN THEOREM

The proof of the theorem proceeds in several steps. Each section in the paper contains a step of the proof. We outline the results of the individual sections and explain how they fit together to prove the theorem.

**Section 3.** We first set the notation and discuss some local and global aspects of minimal surfaces in  $\mathbb{S}^2 \times \mathbb{R}$ . In Theorem 3.2 we describe conformal minimal immersions. We show that a proper annulus is parabolic, and in a conformal parametrization  $X : \mathbb{C}/\tau\mathbb{Z} \rightarrow \mathbb{S}^2 \times \mathbb{R}$ ,  $z \mapsto (G(z), h(z))$  minimality is equivalent to the harmonicity of both  $G$  and  $h$ . Properness implies  $dh \neq 0$  and we can then find a conformal parametrization with constant Hopf differential  $Q = \langle G_z, G_z \rangle (dz)^2 = \frac{1}{4} \exp(i\Theta) (dz)^2$  and linear vertical component  $h(z) = \operatorname{Re}(-ie^{i\Theta/2}z)$ . The metric of the immersion is given by  $ds^2 = \cosh^2 \omega dz \otimes d\bar{z}$ , and the third coordinate of the unit normal vector given by  $n_3 = \tanh \omega$ . We prove that in these special coordinates the real function  $\omega : \mathbb{C} \rightarrow \mathbb{R}$  is a solution of sinh-Gordon equation (1.1) where  $\Delta$  is the Laplacian in the flat Euclidean metric. We conclude by computing  $\omega$  for the Abresch family.

**Section 4.** Due to a theorem by Meeks-Rosenberg [17], the curvature  $K$  of a properly embedded annulus is bounded by a constant depending on the flux  $F_3$  of  $h : A \rightarrow \mathbb{R}$ . If  $|F_3| \geq \epsilon_0$ , there is a constant  $C_1 > 0$  depending on  $\epsilon_0$  with

$$(2.1) \quad |K| \leq C_1(\epsilon_0)$$

If  $\tau \in \mathbb{C}$  is the period of the annulus, and the Hopf differential is  $Q = \frac{e^{i\Theta}}{4}(dz)^2$ , then we show in Lemma 4.1 that  $F_3 = |\tau|$ . Estimate of curvature on parabolic annuli implies (see lemma 4.3)

the existence of a constant  $C_2(\epsilon_0)$  depending only on the lower bound of the flux of the height function  $h : \mathbb{C}/\tau Z \rightarrow \mathbb{R}$  and the upper bound on the absolute Gaussian curvature with

$$(2.2) \quad \text{Area}(A \cap [-t, t]) \leq C_2(\epsilon_0)t.$$

**Section 5.** Since the function  $\omega$  is a solution of the sinh-Gordon equation, we can apply the iteration of Pinkall-Sterling in section 5 to obtain an infinite hierarchy of formal solutions  $u_1, u_2, \dots$  of the linearized sinh-Gordon equation (LSG for short):

$$(2.3) \quad \mathcal{L} u_n = \Delta u_n + u_n \cosh(2\omega) = 0.$$

Finite type (see Definition 5.1) means that the set of such formal solutions forms a finite dimensional vector space in the kernel of  $\mathcal{L}$ . Combining this with the Meeks-Rosenberg curvature and area growth estimate (2.1), this gives us the first step in the proof of our Theorem: Properly embedded minimal annuli are of finite type (Theorem 5.3).

Finite type solutions of the sinh-Gordon equation give rise to algebraic objects which we call potentials. They are defined as follows.

**Definition 2.1.** *The open subset of a  $3g + 1$  dimensional real vector space of matrices called potentials is*

$$\mathcal{P}_g = \left\{ \xi_\lambda = \sum_{d=-1}^g \hat{\xi}_d \lambda^d \mid \hat{\xi}_{-1} \in \begin{pmatrix} 0 & i\mathbb{R}^+ \\ 0 & 0 \end{pmatrix}, \text{trace}(\hat{\xi}_{-1}\hat{\xi}_0) \neq 0, \right. \\ \left. \hat{\xi}_d = -\bar{\hat{\xi}}_{g-1-d}^t \in \mathfrak{sl}_2(\mathbb{C}) \text{ for } d = -1, \dots, g \right\}.$$

There is a correspondence between finite type solutions of the sinh-Gordon equation and solutions  $\zeta_\lambda : \mathbb{C} \rightarrow \mathcal{P}_g$  of a Lax equation

$$(2.4) \quad d\zeta_\lambda(z) + [\alpha_\lambda(z), \zeta_\lambda(z)] = 0 \text{ with } \zeta_\lambda(0) = \xi_\lambda \text{ where} \\ \alpha(\omega) = \frac{1}{4} \begin{pmatrix} 2\omega_z & i\lambda^{-1}e^\omega \\ i\gamma e^{-\omega} & -2\omega_z \end{pmatrix} dz + \frac{1}{4} \begin{pmatrix} -2\omega_{\bar{z}} & i\bar{\gamma}e^{-\omega} \\ i\lambda e^\omega & 2\omega_{\bar{z}} \end{pmatrix} d\bar{z}$$

The Lax equation ensures that  $\det \zeta_\lambda(z) = \det \xi_\lambda$  does not depend on the variable  $z$ . In Definition 5.6 we define for a potential  $\xi_\lambda = \zeta_\lambda(0)$  the polynomial  $a(\lambda) = -\lambda \det \xi_\lambda$  and the spectral curve  $\Sigma$  as the 2-point compactification of

$$(2.5) \quad \Sigma^* = \{(\nu, \lambda) \in \mathbb{C}^2 \mid \det(\nu \text{Id} - \zeta_\lambda) = 0\} = \{(\nu, \lambda) \in \mathbb{C}^2 \mid \nu^2 = \lambda^{-1}a(\lambda) = -\det \xi_\lambda\}$$

We note that the points over  $\lambda = 0$  and  $\lambda = \infty$  are branch points of  $\Sigma$ . Here  $a$  is a polynomial of degree  $2g$  obeying the reality condition

$$(2.6) \quad |a(0)| = \frac{1}{16}, \quad \lambda^{2g} \overline{a(\bar{\lambda}^{-1})} = a(\lambda) \text{ and } \frac{a(\lambda)}{\lambda^g} \leq 0 \text{ for all } \lambda \in \mathbb{S}^1.$$

**Section 6. Construction of a minimal surface with potential  $\xi_\lambda$ .** We identify the sphere  $\mathbb{S}^2$  with  $\text{SU}_2/U(1)$  and  $(x, y, z) \in \mathbb{R}^3$  with the matrix  $M = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$ . The Lie group  $\text{SU}_2$  (with associated Lie algebra  $\mathfrak{su}_2$ ), acts on  $\mathfrak{su}_2 \cong \mathbb{R}^3$  isometrically by  $M \rightarrow gMg^{-1}$  with  $|M|^2 = \det M$ . The map  $g \rightarrow g\sigma_3^t \bar{g}$  maps  $\text{SU}_2$  into  $\mathbb{S}^2 \subset \mathbb{R}^3$  with  $\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ . An extended frame  $F : \Omega \rightarrow \text{SU}_2$  associate to a map  $G : \Omega \rightarrow \mathbb{S}^2$  is defined by

$$G = F\sigma_3 F^{-1}.$$

To obtain a minimal surface we first solve for an extended frame  $F_\lambda : \Omega \rightarrow \text{SL}_2(\mathbb{C})$

$$F_\lambda^{-1} dF_\lambda = \alpha(z) \text{ with } F_\lambda(0) = \text{Id}$$

for parameter  $\lambda \in \mathbb{C}$ . For  $\lambda \in \mathbb{S}^1$ , we have  $F_\lambda \in \text{SU}_2$  (see section 6). We choose to evaluate at  $\lambda = 1$ . We say then that we 'parametrize the immersion by its Sym point' (see Proposition

A.1). We apply a construction of Sym-Bobenko type which gives us the minimal immersion at  $\lambda = 1$ :

$$(2.7) \quad X(z) = (F_1(z)\sigma_3 F_1^{-1}(z), \operatorname{Re}(-ie^{i\Theta/2}z)) \text{ and } Q_1 = -4\beta_{-1}\gamma_0(dz)^2 = \frac{1}{4}e^{i\Theta}(dz)^2.$$

We remark that by formal computation if we set

$$(2.8) \quad \zeta_\lambda(z) := F_\lambda^{-1}(z)\xi_\lambda F_\lambda(z),$$

we obtain a polynomial killing field  $\zeta_\lambda : \mathbb{C} \rightarrow \mathcal{P}_g$  solution of the Lax equation

$$d\zeta_\lambda + [F_\lambda^{-1}dF_\lambda, \zeta_\lambda] = 0 \text{ and } \zeta_\lambda(0) = \xi_\lambda.$$

Conversely from  $\xi_\lambda$ , we solve the Lax equation to obtain the polynomial killing field  $\zeta_\lambda(z)$ . Now we apply the Iwasawa method to find a map  $F_\lambda : \mathbb{C} \rightarrow \operatorname{SL}_2(\mathbb{C})$  which satisfies equation (2.8), and which in turn gives us the immersion. The symmetry of the matrices  $\xi_\lambda \in \mathcal{P}_g$  contains the information of  $F_\lambda \in \operatorname{SU}_2$  for  $\lambda \in \mathbb{S}^1$ .

Consequently we shall characterize in the remaining sections those potentials  $\xi_\lambda$ , which induce for  $\lambda = 1$  properly embedded minimal annuli in  $\mathbb{S}^2 \times \mathbb{R}$ .

**Section 7. Isospectral set and spectral data.** The condition on a potential  $\xi_\lambda$  to induce a periodic immersion turns out to be a condition on the corresponding polynomial  $a(\lambda)$  and the spectral curve (2.5). Therefore we investigate the set  $I(a)$  (2.9) of all potentials with the same spectral curve:

$$(2.9) \quad I(a) := \left\{ \xi_\lambda \in \mathcal{P}_g \mid \lambda \det \xi_\lambda = -a(\lambda) \text{ and } \beta_{-1}\gamma_0 = a(0) = -\frac{1}{16}e^{i(1-g)\theta} := -\frac{1}{16}e^{i\Theta} \right\}$$

**Curvature estimate on finite type annuli.** The compactness of the isospectral set  $I(a)$  implies an estimate of curvature. This comes from the fact that  $\zeta_\lambda : \mathbb{C} \rightarrow I(a)$  depends on  $\omega$  and its higher derivatives. We bound these derivatives by the modulus of the roots of  $a(\lambda)$ .

**Proposition 2.2.** *If  $\omega : \mathbb{C} \rightarrow \mathbb{R}$  is the solution of the sinh-Gordon equation of a minimal immersion  $X : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$ , then the curvature of  $X$  is equal to*

$$(2.10) \quad K = \tanh^2 \omega - \frac{|\nabla \omega|^2}{\cosh^4 \omega}.$$

For polynomials  $a$  obeying (2.6) the finite type immersions corresponding to all potentials  $\xi_\lambda \in I(a)$  have bounded curvature, if and only if the roots of  $a$  are bounded away from  $\infty$  and 0.

*Proof.* Let  $\bar{\mathcal{M}}^g$  denote the subspace of all complex polynomials  $a$  of degree  $2g$  obeying

$$|a(0)| \neq 0, \quad \lambda^{2g} \overline{a(\bar{\lambda}^{-1})} = a(\lambda) \text{ and } \lambda^{-g}a(\lambda) \leq 0 \text{ for all } \lambda \in \mathbb{S}^1$$

(compare with (2.6)). We prove in Proposition 3.3 [8] that the map

$$(2.11) \quad a : \mathcal{P}_g \rightarrow \bar{\mathcal{M}}^g, \quad \xi \mapsto -\lambda \det \xi_\lambda$$

is a proper map. The Laurent coefficients of  $\xi_\lambda = \sum_{d=-1}^g \lambda^d \hat{\xi}_d$  are

$$\hat{\xi}_d = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \lambda^{-d} \xi_\lambda \frac{d\lambda}{\lambda}.$$

Using a norm

$$\|\hat{\xi}_d\| \leq \frac{1}{2\pi i} \int_{\mathbb{S}^1} \|\lambda^{-d} \xi_\lambda\| \frac{d\lambda}{\lambda} \leq \sup_{\lambda \in \mathbb{S}^1} \sqrt{-\lambda^{-g}a(\lambda)}.$$

Thus each entry  $\hat{\xi}_d$  of  $\xi_\lambda$  is bounded, if  $\sqrt{-\lambda^{-g}a(\lambda)}$  is bounded on  $\mathbb{S}^1$ . For polynomials  $a$  obeying (2.6) this follows from the roots of  $a$  being bounded. Since  $\hat{\xi}_{-1}$  is an upper triangular matrix with coefficient  $4\beta_{-1} = ie^\omega$  (see lemma 6.1 and remark 6.2), the bound of the Laurent

coefficient  $\hat{\xi}_{-1}$  on  $I(a)$  is equivalent to a uniform bound on  $\omega$ . Schauder estimates bound the  $\mathcal{C}^{k,\alpha}$  estimate of  $\omega$  on  $\mathbb{R}/\tau\mathbb{Z} \times \mathbb{R}$ .

Conversely let the curvature of the immersions corresponding to all  $\xi_\lambda \in I(a)$  be bounded. For any roots  $\alpha_1, \dots, \alpha_g$  of  $a$ , such that  $\bar{\alpha}_1^{-1}, \dots, \bar{\alpha}_g^{-1}$  are the remaining roots of  $a$ , there exist an off-diagonal  $\xi \in I(a)$  with

$$\alpha = 0 \quad \beta = \frac{i}{4\lambda\sqrt{\prod_d |\alpha_d|}} \prod_d (1 - \bar{\alpha}_d \lambda) \quad \gamma = \frac{i}{4\sqrt{\prod_d |\alpha_d|}} \prod_d (\lambda - \alpha_d).$$

The corresponding  $\omega$  at  $z = 0$  is due to proposition 6.1 and remark 6.2 equal to  $\omega(0) = -\frac{1}{2} \sum_d \ln |\alpha_d|$  and  $\nabla \omega(0) = 0$ . If the curvature of all these  $\xi_\lambda$  is bounded, then all roots of  $a$  are bounded away from  $\infty$  and 0.  $\square$

**Group action.** We define via the Iwasawa decomposition a commuting group action

$$\pi : \mathbb{C}^g \times I(a) \rightarrow I(a)$$

on the isospectral set. It integrates the family of solutions of the linearized sinh-Gordon equation  $u_1, u_2, \dots$  into deformations of the metric  $\omega$  and then deformations of the extended frame  $F_\lambda$ . The first solution  $u_1 = \omega_z$  integrates as translation on the annulus. The corresponding group action transforms the annulus as a two-dimensional subgroup in the isospectral set. For  $(z, 0, \dots, 0) \in \mathbb{C}^g$ , we have

$$\zeta_\lambda(z) = \pi(z, 0, \dots, 0) \xi_\lambda$$

the polynomial killing field  $\zeta_\lambda : \mathbb{C} \rightarrow \mathcal{P}_g$ .

An important property of spectral curves without singularities i.e. the polynomial  $a(\lambda)$  has only simple roots, is that the isospectral set  $I(a)$  has only one orbit diffeomorphic to a real  $g$ -dimensional torus which is isomorphic to the real part of the Picard group (the real part of the Jacobian of  $\Sigma$ ). This property implies that all annuli related to a regular spectral curve have a quasi-periodic polynomial Killing field as a flat annulus immersed in  $(\mathbb{S}^1)^g$  (see corollary 7.8). Since this polynomial killing field depends only on  $\omega$ , this means that also the metric is quasi-periodic.

**Remark 2.3.** *On minimal annuli of  $\mathbb{S}^2 \times \mathbb{R}$  parameterized by the third coordinate  $h(z) = y$ , the function  $u = \cosh^2 \omega(\partial_x k_g) = (\omega_{xy}) - \tanh \omega(\omega_x)(\omega_y)$  is a Jacobi field. In this expression  $k_g$  is the geodesic curvature (see theorem 3.3) of the horizontal curve. Integrating this normal Jacobi field on the surface gives a variation on  $\omega$  which is  $u_2 = \omega_{zzz} - 2\omega_z^3$ , the second flow in the hierarchy. To integrate this Jacobi field it suffices to look for the second group action on  $\xi_\lambda$  and*

$$\zeta_\lambda(z, t) = \pi(z, t, 0, \dots, 0) \xi_\lambda = F_\lambda(z, t)^{-1} \xi_\lambda F_\lambda(z, t)$$

*is the Killing field which integrates the Shiffman Jacobi field on the surface with extended frame  $F_\lambda(z, t)$ . This integrates the Shiffman Jacobi field in a long time existence and proves that this flow is quasi-periodic.*

For polynomials  $a$  with higher order roots, the isospectral action is not transitive. In this case the isospectral set decomposes into several orbits with respect to the action. Besides one smallest orbit all orbits contain what we call bubbletons. We treat this case in Appendix B. Due to Corollary B.4 there always exists one largest orbit, whose elements  $\xi_\lambda \in I(a)$  have no roots in  $\lambda \in \mathbb{C}^\times$  (see below).

The closing condition is encoded in the spectral curve. If  $\tau$  is the period for an embedded annulus induced by a potential  $\xi_{0,\lambda}$ , then any potential  $\xi_\lambda = \pi(t) \xi_{0,\lambda}$  induces an embedded annulus with the same period  $\tau$ . This means that  $\pi(\tau) \xi_\lambda = \xi_\lambda$  is a trivial action for any  $\xi_\lambda$  in the same orbit as  $\xi_{0,\lambda}$ . The eigenvalue  $\mu(\lambda)$  of  $\lambda \rightarrow F_\lambda(\tau)$  is a holomorphic function on  $\Sigma^*$  with two essential singularities at the branch points over  $\lambda = 0$  and  $\lambda = \infty$  of  $\Sigma$ . At these points the 1-form  $d \ln \mu$

has second order poles with no residue and  $d \ln \mu - \frac{1}{4} i \tau e^{i\Theta} d\sqrt{\lambda}^{-1}$  extends holomorphically at  $\lambda = 0$ . A similar expression at  $\lambda = \infty$  holds. This formula means that the third coordinate of the flux is encoded at the essential singularities of  $\mu$  (see proposition 7.9, 7.10 and definition 7.11). The meromorphic differential takes the form

$$d \ln \mu = \frac{b d\lambda}{\nu \lambda^2}$$

for some polynomial  $b(\lambda)$  of degree  $g + 1$  where  $b(0) = -\frac{1}{2} \tau e^{i\Theta} \in e^{i\Theta/2} \mathbb{R}$ . We call  $(a, b)$  the spectral data of the annulus, and give in Corollary 7.12 a complete characterization of spectral data of periodic immersions.

**Section 8. Isospectral set and embedded annuli.** By the maximum principle at infinity, an embedded annulus has an embedded tubular neighborhood (see lemma 4.3). Using this property and maximum principle at an interior point, we prove in propositions 8.1 and 8.2 that the isospectral action preserves embeddedness in the isospectral set  $I(a)$ . If we have  $\hat{\xi}_\lambda = \pi(t)\xi_\lambda$  and  $\xi_\lambda$  induces an embedded annulus  $X(z, \xi_\lambda)$ , then the potential  $\hat{\xi}_\lambda$  induces an embedded annulus  $X(z, \hat{\xi}_\lambda)$ . We provide first a proof of this property when  $a(\lambda)$  has only simple roots. This implies that if  $\xi_\lambda \in I(a)$  induces an embedded annulus then the whole isospectral set  $I(a)$  has potentials all inducing embedded annuli. In the case where  $a(\lambda)$  has higher order roots, it can happen in general that only few orbits contains potentials inducing embedded annuli.

**Isospectral set for polynomial  $a(\lambda)$  having higher order roots.** Different  $\xi_\lambda$  of different isospectral sets may give the same extended frame  $F_\lambda$ . This is the case in particular if an initial value  $\xi_\lambda$  has a root at some  $\lambda = \alpha_0 \in \mathbb{C}^*$ . Then also the corresponding polynomial Killing field  $\zeta_\lambda(z)$  has a root at  $\lambda = \alpha_0$  for all  $z \in \mathbb{C}$ . In this case we may reduce the order of  $\xi_\lambda$  and  $\zeta_\lambda(z)$  without changing the corresponding extended frame  $F_\lambda$ .

This configuration corresponds to a singular spectral curve i.e. the polynomial  $a(\lambda)$  has a root of order at least two at  $\alpha_0$ . We can remove this singularity without changing the surface. There is a polynomial  $p(\lambda)$  such that  $\tilde{\xi}_\lambda = \xi_\lambda/p$  does not vanish at  $\alpha_0$  and is the initial value of a polynomial Killing field  $\tilde{\zeta}_\lambda(z)$  without zeroes at  $\alpha_0$ . We show in Proposition 4.3 [8] that both polynomial Killing fields  $\zeta_\lambda$  and  $\zeta_\lambda/p$  give rise to the same extended frame  $F_\lambda$ .

**Proposition 2.4.** [8] *If a polynomial Killing field  $\zeta_\lambda$  with initial value  $\xi_\lambda \in I(a) \subset \mathcal{P}_g$  has zeroes in  $\lambda \in \mathbb{C}^*$ , then there is a polynomial  $p(\lambda)$ , such that the following conditions hold:*

- (1)  $\xi_\lambda/p \in \mathcal{P}_{g-\deg p}$  has no zeroes in  $\lambda \in \mathbb{C}^*$  and gives rise to the polynomial Killing field  $\zeta_\lambda/p$ .
- (2) If  $F_\lambda$  and  $\tilde{F}_\lambda$  are the unitary decomposition factors of  $\zeta_\lambda$  and  $\zeta_\lambda/p$  respectively then the immersion parameterized by its Sym point satisfies

$$X(z) = (F_1(z)\sigma_3 F_1^{-1}(z), \operatorname{Re}(-ie^{i\theta_0} z)) = (\tilde{F}_1(p(0)z)\sigma_3 \tilde{F}_1^{-1}(p(0)z), \operatorname{Re}(-ie^{i\theta_0} z))$$

Hence amongst all polynomial Killing fields that give rise to a minimal surface of finite type there is one of smallest possible degree (without adding further poles).

**Proposition 2.5.** [8] *Let  $I(a)$  be the isospectral set associated to a polynomial  $a(\lambda)$ , which satisfies the reality condition.*

- (1) *If  $a(\lambda)$  has a double root  $\alpha_0$  with  $|\alpha_0| = 1$ , then  $I(a) = \{\xi_\lambda \in I(a) \mid \xi_{\alpha_0} = 0\}$  and there is an isomorphism*

$$I(a) \longrightarrow I\left(\frac{\alpha_0 a}{(\lambda - \alpha_0)^2}\right) \text{ defined by } \xi_\lambda \mapsto \frac{\sqrt{\alpha_0} \xi_\lambda}{\lambda - \alpha_0}$$



- (2) If  $a(\lambda)$  has double root  $\alpha_0$  with  $|\alpha_0| \neq 1$  then  $I(a) = \{\xi_\lambda \in I(a) \mid \xi_{\alpha_0} \neq 0\} \cup \{\xi_\lambda \in I(a) \mid \xi_{\alpha_0} = 0\}$  and there is an isomorphism

$$\{\xi_\lambda \in I(a) \mid \xi_{\alpha_0} = 0\} \longrightarrow I\left(\frac{a}{(\lambda - \alpha_0)(1 - \bar{\alpha}_0\lambda)}\right) \text{ defined by } \xi_\lambda \mapsto \frac{1}{(\lambda - \alpha_0)(1 - \bar{\alpha}_0\lambda)} \xi_\lambda.$$

*Proof.* (1) If  $a$  has a double root at  $\alpha_0$  with  $|\alpha_0| = 1$ , then for any  $\xi_\lambda \in I(a)$ , we have  $\xi_{\alpha_0} = 0$ , because the determinant is a norm for any  $\lambda \in \mathbb{S}^1$ .

(2) If  $a$  has a double root at  $\alpha_0$ , with  $|\alpha_0| \neq 0$ , then the isospectral set splits into a part which contain potentials with a zero at  $\alpha_0$  (and we can remove again the singularity), and the set of potential not zero at  $\alpha_0$ . But in this last case, this mean that  $\xi_{\alpha_0}$  is a nilpotent matrix, and we say that the surface has a bubbleton. We study this case in Appendix B and in the next subsection.  $\square$

**Remove or add bubbletons on annulus.** Bubbletons occur when the polynomial  $a(\lambda)$  has higher order roots  $\alpha_1, \dots, \alpha_k$ . We can assume (repeating several times the same roots if necessary) that

$$a(\lambda) = \prod_{i=1}^k (\lambda - \alpha_i)^2 (1 - \lambda \bar{\alpha}_i)^2 \tilde{a}(\lambda) = (\lambda - \alpha_j)^2 (1 - \lambda \bar{\alpha}_j)^2 \tilde{a}_j(\lambda)$$

where  $\tilde{a}(\lambda)$  has only simple roots,  $\deg a(\lambda) = 2g$  and  $\deg \tilde{a}(\lambda) = 2g - 2k$ . The group action

$$\pi(\cdot)\xi_\lambda : \mathbb{C}^g \rightarrow I(a)$$

preserves the degree of roots of  $\xi_\lambda$  at  $\alpha_1, \dots, \alpha_k$  (if  $\xi_{0,\alpha_i} = 0$ , then  $\xi_{\alpha_i}(t) = \pi(t)\xi_{0,\alpha_i} = 0$ ). Hence the isospectral action defines several orbits in  $I(a)$ . The group  $\mathbb{C}^g$  acts on  $I(a)$  but the stabilizer  $\Gamma_{\xi_\lambda} = \{t \in \mathbb{C}^g \mid \pi(t)\xi_\lambda = \xi_\lambda\}$  is no longer isomorphic to  $\mathbb{Z}^{2g}$ .

If  $\xi_{0,\lambda} \in I(a)$  and  $O = \{\xi_\lambda \in I(a) \mid \pi(t)\xi_{0,\lambda} = \xi_\lambda \text{ for } t \in \mathbb{C}^g\}$  is the orbit of some potential  $\xi_{0,\lambda}$  having roots at  $\alpha_j$ , there is a real two-dimensional subgroup  $E^2 \subset \mathbb{C}^g$  which acts trivially on  $\xi_{0,\lambda}$ . In this case  $O$  is diffeomorphic to  $I(\tilde{a}_j)$ . We can remove the roots of  $\xi_{0,\lambda}$  without changing the immersion given by  $F_\lambda$ . If we can do that for any higher order roots of  $a$ , we obtain by removing all zeroes a quasi-periodic annulus induced by a potential  $\tilde{\xi}_\lambda \in I(\tilde{a}) \cong (\mathbb{S}^1)^{2(g-k)}$ .

Now, we consider the orbit  $O$  of a potential  $\xi_{0,\lambda} \in I(a)$ , where  $\xi_{0,\alpha_i} \neq 0$  and  $\alpha_i$  is a higher order root of  $a(\lambda) = (\lambda - \alpha_i)^2 (1 - \lambda \bar{\alpha}_i)^2 \tilde{a}_i(\lambda)$ . If  $\xi_{\alpha_i} \neq 0$  and  $\det \xi_{\alpha_i} = 0$ , then the matrix is nilpotent and defines a complex line  $L' \in \mathbb{CP}^1$  related to the one dimensional complex subspace  $\ker \xi_{0,\alpha_i} = \text{Im } \xi_{0,\alpha_i}$  (see appendix A). We can decompose uniquely the potential as

$$\xi_{0,\lambda} = (L', \tilde{\xi}_{0,\lambda}) = p(\lambda) h_{L',\alpha_i} \tilde{\xi}_{0,\lambda} h_{L',\alpha_i}^{-1}$$

where  $\tilde{\xi}_{0,\lambda} \in I(\tilde{a}_i)$  has a determinant with a lower degree  $\alpha_i$ , and  $h_{L',\alpha_i}(\lambda) \in \Lambda_r^+ \text{SL}_2(\mathbb{C})$  (see r-Iwasawa decomposition with  $r < |\alpha_i| < 1$ ) is a matrix depending on parameter  $\lambda$  with pole at  $\alpha_i$  and  $1/\bar{\alpha}_i$  and  $h_{L',\alpha_i}(0)$  is an upper triangular matrix. A property of this decomposition is that  $\xi_{0,\alpha_i} = 0$  if and only if the eigenline  $(L')^\perp$  is an eigenline of  $\tilde{\xi}_{0,\alpha_i}$ .

There is a two dimensional subgroup action  $\tilde{\pi} : \mathbb{C} \times I(a) \rightarrow I(a)$  which preserves the second factor of the decomposition  $(L', \tilde{\xi}_{0,\lambda}) \in \mathbb{CP}^1 \times I(\tilde{a}_j)$  and acts transitively on the first factor  $L' \in (\mathbb{CP}^1)^\times$ . The set  $(\mathbb{CP}^1)^\times$  is  $\mathbb{CP}^1$  minus the fixed point of the action  $\tilde{\pi}$ . If  $(L')^\perp$  is an eigenline of  $\tilde{\xi}_{\alpha_0}$ , the potential  $\xi_\lambda$  has a zero at  $\lambda = \alpha_0$  and the subgroup  $\tilde{\pi}$  acts trivially. This decomposition implies that the orbit  $O$  is dense in  $I(a)$ . The differential structure of  $O$  comes from the differentiability of the action away from its fixed point where  $\tilde{\pi}$  extends only continuously.

Since the action preserves embeddedness (see proposition 8.2 and B.2), there is a continuous isospectral deformation of embedded annuli  $A(t)$ , induced by potentials  $\xi_\lambda(t) = \tilde{\pi}(\beta(t))\xi_{0,\lambda}$ , such that  $L'(\beta(t)) \rightarrow L'_0$  with  $(L'_0)^\perp$  are eigenlines of  $\tilde{\xi}_{0,\alpha_i}$ .

At the limit we have an annulus induced by  $\xi_{1,\lambda} = p(\lambda)h_{L'_0,\alpha_0}\tilde{\xi}_{0,\lambda}h_{L'_0,\alpha_0}^{-1}$  which is the limit of embedded annuli obtained from  $\xi_\lambda(t) = \tilde{\pi}(\beta(t))\xi_{0,\lambda} = p(\lambda)h_{L'(\beta(t)),\alpha_0}\tilde{\xi}_{0,\lambda}h_{L'(\beta(t)),\alpha_0}^{-1}$ . The potential  $\xi_{1,\lambda}$  induces an embedded annulus but has a zero at the roots. Then we can simplify the potential by the polynomial  $p(\lambda)$  and decrease the spectral arithmetic genus of  $\Sigma$ . Doing this process inductively we can remove all higher order roots of  $a(\lambda)$ , one by one.

As a result the classification of all embedded minimal annuli relies on two problems:

**Problem A.** Classify all spectral data  $(a, b)$  with  $a$  having only simple roots and  $I(a)$  corresponding to embedded minimal annuli.

**Problem B.** Determine for all such spectral data  $(a, b)$  all possibilities to add bubbletons without destroying the embeddedness.

**Compactness of the space of embedded annuli and spectral data.** We consider the space of embedded annuli of finite type with bounded spectral genus  $g \leq g_0$  and flux bounded away from 0 with the topology of the uniform convergence on compact set:

$$\mathcal{M}_{g_0}^{emb}(\epsilon_0) = \{X : \mathbb{C}/\tau\mathbb{Z} \rightarrow \mathbb{S}^2 \times \mathbb{R}; |\tau| \geq \epsilon_0, X \text{ embedded}, (a, b, \xi_\lambda) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda] \times \mathcal{P}_g, g \leq g_0\}$$

**Theorem 2.6.** *A sequence  $(X_n)_{n \in \mathbb{N}} \in \mathcal{M}_{g_0}^{emb}(\epsilon_0)$  has a subsequence converging to  $X_0 \in \mathcal{M}_{g_0}^{emb}(\epsilon_0)$  and there is a constant  $C_3 > 0$  such that  $|\tau| \leq C_3$  on  $\mathcal{M}_{g_0}^{emb}(\epsilon_0)$ .*

*Proof.* We consider a sequence of embedded minimal annuli  $(X_n)_{n \in \mathbb{N}}$  with associate spectral data  $(a_n, b_n, \xi_{\lambda,n})$ . Since we assume that  $|\tau_n| \geq \epsilon_0$  is bounded away from zero, the curvature is bounded and  $X_n$  is locally a graph over a disc of radius  $\delta > 0$ . There is a subsequence of immersions  $X_{n_k} : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  which converge on compact sets to  $X_0$  a minimal surface. By the maximum principle the annulus  $X_0$  is embedded. The surface  $X_0 : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  is an annulus if we bound the period  $|\tau|$  above. By lemma 4.3, the linear area growth satisfies

$$\text{Area}(X_n \cap \mathbb{S}^2 \times [-t, t]) \leq C_2(\epsilon_0)t.$$

On the other hand, using the fact that the metric is  $ds^2 = \cosh^2 \omega_n |dz|^2$ , we conclude that  $2t|\tau_n| \leq C_3 \text{Area}(X_n([0, \tau_n] \times [-t, t]))$ , and thus the flux is uniformly bounded on the space of embedded annuli  $\mathcal{M}_{g_0}^{emb}(\epsilon_0)$ .

The sequence of immersions  $X_{n_k}$  is induced by a sequence of potentials  $\xi_{\lambda,n_k} \in \mathcal{P}_{g_k}$  with  $g_k \leq g_0$ . The bound on the curvature of proposition 2.2 implies that all roots of the polynomials  $a_n$  are bounded away from 0 and  $\infty$  for any  $n \in \mathbb{N}$ . Since  $I(a_n)$  are compact sets, there is a subsequence of potentials  $\xi_{\lambda,n_k}$  which converge to a potential  $\xi_{\lambda,0} \in I(a_0)$ .

The potential depends on the metric  $\omega_{n_k}$  and its higher derivatives at one point. By elliptic theory  $\omega_{n_k} \rightarrow \omega_0$  in the  $\mathcal{C}^{k,\alpha}$  topology. This implies that the sequence of solutions of LSG  $u_1, u_2, \dots$  converges on any compact set. After renormalization of coefficients, the sequence of algebraic relations

$$\sum_{i=1}^{g=g_k} a_{i,n_k} u_{i,n_k} + b_{i,n_k} \bar{u}_{i,n_k} = 0$$

converge to some similar algebraic condition on the solution  $u_1, u_2, \dots$  of the metric  $\omega_0$ . The spectral genus of  $X_0$  is bounded by  $g_0$  and the sequence of potentials  $\xi_{\lambda,n_k}$  converge to a potential  $\xi_{\lambda,0}$  which induces the embedded annulus  $X_0$  with flux  $\epsilon_0 \leq |\tau| \leq C_3$ . Since the limit is an annulus of period  $\tau_0$ , the monodromy eigenvalue  $\mu$  is well defined and there is a subsequence of spectral data  $(a_{n_k}, b_{n_k})$  which converge to some  $(a_0, b_0) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ .  $\square$

**Deformation of spectral data preserves embeddedness.** We consider a potential  $\xi_{\lambda,0}$  which induces an embedded annulus  $X_0(z)$  with spectral data  $(a_0, b_0)$  and a sequence of annuli  $(X_n(z))_{n \in \mathbb{N}}$  which converge to  $X_0(z)$  on compact sets. We denote by  $(a_0, b_0)$  and  $(a_n, b_n)$  the corresponding spectral data, and by  $\xi_{\lambda,0}, \xi_{\lambda,n}$  the corresponding potentials with polynomial Killing field  $\zeta_{\lambda,0}(z), \zeta_{\lambda,n}(z)$ . We assume that  $a_n \rightarrow a_0, b_n \rightarrow b_0, \tau_n \rightarrow \tau_0$  and  $\xi_{\lambda,n} \rightarrow \xi_{\lambda,0}$ .

**Proposition 2.7.** *If  $X_0(z)$  is a finite type embedded minimal annulus associated to an isospectral set having only one orbit (i.e.  $a_0(\lambda)$  has only simple roots or removable double roots), then there is  $n_0 > 0$  such that for  $n \geq n_0$ , the annuli  $X_n$  are embedded.*

*Proof.* If  $X_0(z)$  is embedded then we prove that  $X_n(z)$  is an embedded annulus for  $n$  large enough. Assume the contrary, that there is a subsequence of non embedded annuli  $X_{n_k}$  which converge to  $X_0$  on compact sets and there is a diverging sequence of real numbers  $y_k$  such that  $x \rightarrow X_{n_k}(x, y_k)$  is a self-intersecting horizontal curve in  $\mathbb{S}^2$ .

The sequence of potentials  $\pi(\mathbf{i}y_k, 0, \dots, 0)\xi_{\lambda, n_k}$  induce annuli  $\tilde{X}_{n_k}(x, y) = X_{n_k}(x, y + y_k)$ . The sequence  $\pi(\mathbf{i}y_k, 0, \dots, 0)\xi_{\lambda, n_k}$  has a subsequence converging to an element  $\hat{\xi}_{\lambda,0}$  of  $I(a_0)$  since  $a_{n_k} \rightarrow a_0$  and  $I(a_n)$  are compact subsets of  $\mathcal{P}_g$  for any  $n$ . Since the action is transitive on  $I(a_0)$ , there is  $t \in \mathbb{C}^g$  such that  $\hat{\xi}_{\lambda,0} = \pi(t)\xi_{\lambda,0}$  and  $\hat{\xi}_{\lambda,0}$  induces an embedded annulus by proposition 8.1.

The Iwasawa decomposition is a real analytic diffeomorphism and  $(t, \xi_\lambda) \rightarrow \pi(t)\xi_\lambda$  is uniformly continuous on compact sets from  $\mathbb{C}^g \times \mathcal{P}_g$  to  $\mathcal{P}_g$ . Hence the immersion  $x \rightarrow \tilde{X}_{n_k}(x, 0)$  for  $x \in [0, \tau_{n_k}]$  is smoothly converging to  $x \rightarrow \hat{X}(x, 0)$  for  $x \in [0, \tau_0]$ , a contradiction.  $\square$

**Proposition 2.8.** *If  $X_0(z)$  is an embedded simple bubbleton with  $a_0(\lambda)$  having only one higher order root (i.e.  $a_0(\lambda) = (\lambda - \alpha_0)^2(1 - \lambda\bar{\alpha}_0)^2\tilde{a}_0(\lambda)$  and  $\tilde{a}_0(\lambda)$  has only simple roots), then there is  $n_0 > 0$  such that for  $n \geq n_0$ ,  $X_n$  are embedded annuli.*

*Proof.* In this case  $I(a_0) = \{\xi_\lambda \in I(a_0) \mid \xi_{\alpha_0} = 0\} \cup \{\xi_\lambda \in I(a_0) \mid \xi_{\alpha_0} \neq 0\}$  has two orbits, and we apply the same argument. We only have to prove that if  $\xi_\lambda \in I(a_0)$ , then  $\xi_\lambda$  induces an embedded annulus.

Since the bubbleton is embedded, we proved in proposition that any  $\tilde{\pi}(\beta)\xi_{\lambda,0} = (L'(\beta), \tilde{\xi}_\lambda)$  induce an embedded bubbleton. Applying the removing bubbleton procedure we can find an isospectral action which ends in the limit in a quasi-periodic annulus induced by  $\tilde{\xi}_{\lambda,0} \in I(\tilde{a}_0)$ . Since the action extends continuously at the fixed point,  $\tilde{\xi}_{\lambda,0}$  induces an embedded annulus. Hence we conclude that there is a potential in each orbit which induces an embedded annulus. Then  $I(a_0)$  is a set of potentials which induce embedded annuli and we can repeat the argument above.  $\square$

**Remark 2.9.** *The sequence  $(a_n, b_n)$  of spectral data can have various limits: (1) It can be a sequence of polynomials with simple roots and two roots of  $a_n$  which coalesce in the limit at  $\alpha_0$ . (2) It can be a sequence of spectral data with a double root  $\alpha_0(s)$  which induce a smooth deformation of bubbletons, and we will use this in the sequel to prove that there are no bubbletons on Abresch annuli.*

**Section 9.** We adapt the Whitham deformation to our setting. Krichever defined this deformation in the study of space moduli theory. This deformation changes the spectral data in such a way that the corresponding annulus flows along integral curves defined on the set of  $(a, b)$ . We consider an embedded annulus with spectral data  $(a, b)$  which satisfies the conditions of definition 7.11. We deform the spectral curve  $\Sigma$  in such a way that the immersion stays closed. We have to find a deformation of  $a(\lambda)$  by moving its roots without destroying the global properties of the holomorphic function  $\mu : \Sigma^* \rightarrow \mathbb{C}$  with prescribed essential singularities at the two marked points  $(\infty, 0)$  and  $(\infty, \infty)$ . To do that we consider the space moduli of spectral data  $\{(a, b)\}$  and we derive vector fields on these data. The integral curves of this vector field

are differentiable families of spectral data of minimal cylinders. This is the content of section 9. We consider a real polynomial  $c(\lambda)$  of degree at most  $g + 1$  with  $\lambda^{g+1} \overline{c(1/\bar{\lambda})} = c(\lambda)$  and the equation

$$\partial_t \ln \mu = \frac{c(\lambda)}{\nu \lambda}.$$

On the other hand we have the condition  $d \ln \mu = \frac{b d\lambda}{\nu \lambda^2}$  and these equations are compatible on the integral curves if and only if

$$-2\dot{b}a + b\dot{a} = -2\lambda ac' + ac + \lambda a'c.$$

This equation defines the values of  $\dot{a}$  at the roots of  $a$  and  $\dot{b}$  at the roots of  $b$ . This is well defined when  $a$  and  $b$  have no common roots. The closing condition of the immersion is preserved if  $c(1) = 0$  (the Sym point stays at  $\lambda = 1$ ) and  $\text{Im}(c(0)/b(0)) = 0$  for the third coordinate to stay periodic (see theorem 9.2).

**Connectedness of the space of embedded minimal cylinders of finite type.** We prove in this section that every connected component of  $\mathcal{M}_{g_0}^{emb}(\epsilon_0)$  has only one connected component which contains Abresch annuli. In the following Theorem we construct a sequence of minimal annuli for which the flux attain its maximum at a flat cylinder.

**Theorem 2.10.** *Every connected component of the space  $\mathcal{M}_{g_0}^{emb}(\epsilon_0)$  contains a sequence  $(X_n)_{n \in \mathbb{N}}$  converging to the flat cylinder. Hence the space  $\mathcal{M}_{g_0}^{emb}(\epsilon_0)$  has only one connected component which contains Abresch annuli.*

*Proof.* Consider a properly embedded minimal annulus  $X : \mathbb{C}/\tau\mathbb{Z} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  associate to spectral data  $(a, b)$ . We consider the connected component of the space  $\mathcal{M}_{g_0}^{emb}(\epsilon_0)$ , which contains  $X$ . Choose in this connected component a sequence of annuli which maximize the flux  $|\tau|$ .

By theorem 2.6, there is a subsequence  $(X_n)_{n \in \mathbb{N}}$  which converges to some  $X_0 \in \mathcal{M}_{g_0}^{emb}(\epsilon_0)$ . Every connected component contains a maximum of the flux  $|\tau|$ . We consider this annulus which is embedded, has spectral data  $(a_0, b_0)$  and induced by a potential  $\xi_{\lambda,0}$  with maximum of the flux of the connected component.

In case when  $\Sigma$  has singularities i.e. that  $a_0(\lambda)$  has higher order roots we remove zeroes of the potential  $\xi_{\lambda,0}$  and then obtain a bubbleton. Now we use the isospectral deformation which is transitive on an orbit of a bubbleton to go further along the orbit and change from one orbit to a lower dimensional orbit as we described above. We apply what we call "remove a bubbleton" to get at the limit an embedded annulus with spectral curve without singularities, and consequently an annulus with quasi-periodic metric  $\omega$ . From Proposition B.3 we are ensured that during this process the period  $\tau$  does not change, as it is an isospectral invariant and we stay in the same connected component by continuity of the isospectral action.

We denote this annulus by  $X_1$  with spectral data  $(a_1, b_1)$  where  $a_1$  has only simple roots. We deform this cylinder and increase the flux if  $X_1$  is not the flat embedded annulus.

Due to Proposition 2.7 the deformation of the spectral data preserves the property that the corresponding isospectral sets correspond to embedded minimal annuli. If  $a$  and  $b$  have no common roots we define a vector field  $c(\lambda)$  over the spectral data  $(a, b)$  preserving the closing condition. We remark that

$$\text{Re} \frac{c(0)}{b(0)} = -2\partial_t \ln |\tau|$$

Then assuming that  $\text{Re} \frac{c(0)}{b(0)} < 0$ , we increase the flux along the deformation. This is possible if we have enough degree of freedom in the choice of the polynomial  $c(\lambda)$  which is the case only if the spectral genus  $g > 0$ . For any local maximum of the flux, the spectral genus is zero, see Lemma 9.1.

In the case where  $a$  and  $b$  have several common roots, we look for a special reparameterization of spectral data  $(a, b)$  by the value of the period map  $\mu$  at its branch points. We prove with the stable manifold theorem that there is a trajectory which moves out of the singularity (see section 9.2 and proposition 9.5). We increase the flux when we pass through this singularity when we move out. Since  $a$  has only simple roots, the elements of  $I(a)$  have no roots on  $\mathbb{C}^*$ . Due to Proposition 4.9 [8] continuous paths of spectral data can be lifted in a neighbourhood of  $(a, b)$  to continuous paths of potentials and annuli.  $\square$

**Section 10. Isolated property of Abresch annuli.** Consider the family  $\mathcal{A}$  of Abresch annuli. If  $\mathcal{M}_{g_0}^{emb}(\epsilon_0) \setminus \mathcal{A} \neq \emptyset$ , we can find a sequence of embedded annuli  $(X_n)_{n \in \mathbb{N}} \in \mathcal{M}_{g_0}^{emb}(\epsilon_0) \setminus \mathcal{A}$  converging to  $X_0 \in \mathcal{A}$  with spectral data  $(a_0, b_0)$  and potential  $\xi_{\lambda,0}$ . By theorem 2.6, the spectral data  $(a_n, b_n, \xi_{\lambda,n})$  has a converging subsequence with limit  $(a, b, \xi_\lambda)$ .

If  $(X_n)$  with spectral data  $(a_n, b_n, \xi_{\lambda,n})$  is not foliated by constant curvature curves then the Shiffman's Jacobi field  $u(n)$  of  $X_n$  has no zero. The idea is to use  $u(n)$  to construct a non zero bounded Jacobi field  $v_0$  on  $X_0$ . Using the four vertex theorem on embedded horizontal curve we conclude that  $v_0$  has at least four zeroes and then  $v_0 \equiv 0$ , a contradiction by lemma 10.1.

Since  $X_n$  converges to  $X_0$  on compact sets,  $u(n)$  converges to zero on compact sets. Since  $\omega(n)$  is bounded by proposition 4.4 on the whole annulus, the Jacobi field  $u(n) = \omega_{xy}(n) - \tanh \omega(n) \omega_x(n) \omega_y(n)$  is bounded on  $X_n$ . We consider  $c_n := \sup_{X_n} |u(n)|$  and

$$v(n) := \frac{u(n)}{c_n}.$$

Then there is a subsequence which converges by Arzela-Ascoli's theorem to a bounded function  $v_0$  on  $X_0$ . If there is a subsequence  $n_k$  such that  $u(n_k)$  takes its maximum value in finite time, then  $v_0$  is not zero since the point where the supremum is attained has an accumulation point  $z_0$  where  $v_0(z_0) = 1$ .

In the case where  $u(n)$  take its maximum value at infinity,  $v_0$  could be zero. Consider the point  $\bar{z}(n) = (\bar{x}(n), \bar{y}(n))$  on the annulus where  $u(n)(\bar{x}(n), \bar{y}(n)) = \frac{1}{2} \sup_{X_n} |u(n)|$  and  $\bar{y}(n) \rightarrow \infty$ . We consider curves  $\gamma(\bar{y}(n)) = X_n \cap \{y = \bar{y}(n)\}$  and we can find a divergent subsequence  $y_{n_k}$ , such that  $X_k = \pi(iy_{n_k}, 0, \dots, 0)X_{n_k}(x, y) = X_{n_k}(x, y + y_{n_k})$  is converging to an annulus  $X_1$ , with spectral data  $(a, b, \xi_{\lambda,1})$  (since  $a_n \rightarrow a$  and  $\tau_n \rightarrow \tau_0$ ).

Since the limit  $X_0$  is geometrically an Abresch annulus, we have that  $a(\lambda) = q(\lambda)a_0(\lambda)$  where

$$q(\lambda) = \prod_{i,j=1}^{i,j=k,l} (\lambda - \alpha_i)^2 (1 - \lambda \bar{\alpha}_i)^2 (\lambda - \beta_j)^2 \text{ with } |\alpha_i| \neq 1, |\beta_j| = 1.$$

**Case 1  $X_1$  is an Abresch annulus.** For example in the case where  $q(\lambda)$  has only roots  $|\beta_j| = 1$  on the unit circle, the corresponding potential  $\xi_{\lambda,1}$  has zeroes at these roots. We can remove these roots without changing the extended frame, and  $X_1$  is a finite translation of the Abresch annulus  $X_0$ . In the general situation we are not able to decide which geometric limit we have.

We parameterize  $X_n$  and  $X_1$  by its third coordinate. We denote by  $ds_n = \cosh^2 \omega_n |dz|^2$  and  $ds_1 = \cosh^2 \omega_1 |dz|^2$  the associated metrics with  $\omega_n \rightarrow \omega_1$  uniformly. Jacobi operators are given by

$$\mathcal{L}_n = \frac{1}{\cosh^2 \omega_n} \left( \partial_x^2 + \partial_y^2 + 1 + 2 \frac{|\nabla \omega_n|^2}{\cosh^2 \omega_n} \right).$$

Since  $\omega_n$  is bounded by Proposition 4.4, the Jacobi field  $u_n = \omega_{n,xy} - \tanh \omega_n \omega_{n,x} \omega_{n,y}$  is bounded on  $X_n$  and

$$v_n := \frac{u_n}{\sup |u_n|}$$

converges by Arzela-Ascoli's theorem to a bounded solution of  $\mathcal{L}_1 v_0 = 0$ , a bounded Jacobi function on  $X_1$ . Since  $u(n) = \cosh^2 \omega_n \partial_x k_g$  (see theorem 3.3), it has at least four zeroes on any level horizontal curve by the four vertex theorem (see [12]). The set of curves  $\Gamma = \{v_n^{-1}(0)\} = \{u_n^{-1}(0)\}$  describes at least four nodal domain intersecting every horizontal curve (see theorem 3.3). By counting the number of zeroes of  $v_n$  on each horizontal section  $x \rightarrow X(x, y_0)$ , we deduce that  $v_0$  cannot have generically two zeroes on horizontal curves.

If not, it means that two or three zeroes of  $v_n$  are collapsing in the limit. We will find an open interval  $y \in ]t_1, t_2[$ , such that on every curve  $\gamma(t) = A \cap \mathbb{S}^2 \times \{t\}$  the zeroes of  $v_n$  are collapsing at the limit in two zeroes. A collapsing of zeroes will produce a new nodal curve  $\Gamma_0 = v_0^{-1}(0)$  generically transverse to horizontal section of  $X_1 \cap \mathbb{S}^2 \times [t_1, t_2]$ . We can find a horizontal section transverse to  $\Gamma_0$ . Since  $\Gamma_0$  is a limit of several nodal curves collapsing together at the limit,  $v_0$  will not change sign along  $\gamma(t)$  crossing  $\Gamma_0$ , or  $v_0$  will change sign but with  $\partial_x v_0 = 0$  on  $\Gamma_0$ . This contradicts a theorem of Cheng [3] on the singularity of nodal curves for the solution of an elliptic operator which are isolated and describing equiangular curves at the singularity.

In summary the four vertex theorem implies that  $v_0$  has at least four zeroes generically on each horizontal section. Now a careful analysis on the operator of  $X_1$  will give a contradiction. We conclude by the lemma 10.1, that a such Jacobi field cannot exist on an Abresch annulus  $X_1$ .

**Case 2  $X_1$  is not an Abresch annulus.** In this case, the polynomial  $a(\lambda)$  has an isospectral set  $I(a)$  with several orbits. This situation occurs when there is a double root  $|\alpha_i| \neq 1$ . If  $X_1$  is not an Abresch annulus, it implies that  $\xi_{\lambda,1}$ , limit of isospectral action of  $\xi_{\lambda,n}$  is in a higher dimensional orbit in  $I(a)$ . The annulus  $X_1$  is a bubbleton on an Abresch annulus.

**Section 11.** Using our 'removing bubbletons' procedure, we can assume the existence of an embedded simple bubbleton on the spectral curve of an Abresch annulus. The idea explained in section 11 follows from two things. One is the possibility to deform a simple bubbleton of spectral genus 1 or 2 to a simple bubbleton on the flat cylinder.

Then we prove that we can open the double point of the bubbleton into a genus two annulus preserving embeddedness. This involves increasing the genus of the spectral curve. We can open a node on  $\Sigma$  keeping the period closed, if the node is on a double point  $\alpha_0$ , where  $\mu(\alpha_0) = \pm 1$ . Opening the node into a higher genus curve preserves embeddedness if and only if all orbits of the isospectral set induce embedded annuli i.e. if the bubbleton is embedded (by hypothesis). We do this by increasing the flux  $|\tau|$ . But since we are on the flat cylinder, the period is already at least  $2\pi$ , and by the above, we are in a component which contains a genus zero example of flux strictly greater than  $2\pi$ , a covering of the flat annulus. This contradicts the embedded property of deforming spectral curve along deformation and finishes the proof of the theorem.

### 3. MINIMAL ANNULI AND THE SINH-GORDON EQUATION

**Local parametrization.** We consider  $X = (G, h) : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  a minimal surface conformally immersed in  $\mathbb{S}^2 \times \mathbb{R}$ . As usual write  $z = x + iy$ . The horizontal component  $G : \mathbb{C} \rightarrow \mathbb{S}^2$  of the minimal immersion is a harmonic map. If we denote by  $(\mathbb{C}, \sigma^2(u)|du|^2)$  the complex plane with metric induced by the stereographic projection of  $\mathbb{S}^2$ , the map  $G$  satisfies

$$G_{z\bar{z}} + 2(\log \sigma \circ G)_u G_z G_{\bar{z}} = 0.$$

The holomorphic *quadratic Hopf differential* associated to the harmonic map  $G$  is given by

$$Q(G) = (\sigma \circ G)^2 G_z \bar{G}_z (dz)^2 := \phi(z)(dz)^2.$$

The function  $\phi$  depends on  $z$ , whereas  $Q(G)$  does not. Conformality reads

$$|G_x|_\sigma^2 + (h_x)^2 = |G_y|_\sigma^2 + (h_y)^2 \quad \text{and} \quad \langle G_x, G_y \rangle_\sigma + (h_x)(h_y) = 0$$

hence  $(h_z)^2(dz)^2 = -Q(G)$ . The zeroes of  $Q$  are double, and we can define  $\eta$  as the holomorphic 1-form  $\eta = \pm 2i\sqrt{Q}$ . The sign is chosen so that

$$h = \operatorname{Re} \int \eta.$$

The unit normal vector  $n$  in  $\mathbb{S}^2 \times \mathbb{R}$  has third coordinate

$$\langle n, \frac{\partial}{\partial t} \rangle = n_3 = \frac{|g|^2 - 1}{|g|^2 + 1} \quad \text{where} \quad g^2 := -\frac{G_z}{G_{\bar{z}}}.$$

We define the real function  $\omega : \mathbb{C} \rightarrow \mathbb{R}$  by

$$n_3 := \tanh \omega.$$

We express the differential  $dG$  independently of  $z$  by:

$$dG = G_{\bar{z}}d\bar{z} + G_z dz = \frac{1}{2\sigma \circ G} \overline{g^{-1}\eta} - \frac{1}{2\sigma \circ G} g \eta$$

and the metric  $ds^2$  is given (see [4]) in a local coordinate  $z$  by

$$ds^2 = (|G_z|_\sigma + |G_{\bar{z}}|_\sigma)^2 |dz|^2 = \frac{1}{4}(|g|^{-1} + |g|)^2 |\eta|^2 = 4 \cosh^2 \omega |Q|.$$

We remark that the zeroes of  $Q$  correspond to the poles of  $\omega$ , so that the immersion is well defined. Moreover the zeroes of  $Q$  are points, where the tangent plane is horizontal. The Jacobi operator is

$$\mathcal{L} = \frac{1}{4|Q| \cosh^2 \omega} (\partial_x^2 + \partial_y^2 + \operatorname{Ric}(n) + |dn|^2)$$

and can be expressed in terms of  $Q$  and  $\omega$  by

$$(3.1) \quad \mathcal{L} = \frac{1}{4|Q| \cosh^2 \omega} \left( \partial_x^2 + \partial_y^2 + 4|Q| + \frac{2|\nabla \omega|^2}{\cosh^2 \omega} \right)$$

Since  $n_3 = \tanh \omega$  is a Jacobi field obtained by vertical translation in  $\mathbb{S}^2 \times \mathbb{R}$ , we have  $\mathcal{L} \tanh \omega = 0$  and

$$\Delta \omega + |Q| \sinh \omega \cosh \omega = 0,$$

where  $\Delta = \partial_x^2 + \partial_y^2$  is the Laplacian of the flat metric.

**Minimal annuli.** Consider a minimal annulus  $A$  properly embedded in  $\mathbb{S}^2 \times \mathbb{R}$ . If  $A$  is tangent to a horizontal section  $x_3 = 0$ , the set  $A \cap \{x_3 = 0\}$  defines on  $A$  a compact component in some half-space  $x_3 \geq 0$  or  $x_3 \leq 0$  with boundary in  $\mathbb{S}^2 \times \{0\}$ , a contradiction to the maximum principle. Hence the annulus is transverse to every horizontal section  $\mathbb{S}^2 \times \{t\}$ . The third coordinate map  $h : A \rightarrow \mathbb{R}$  is a proper harmonic map on each end of  $A$ , with  $dh \neq 0$ . Then each end of  $A$  is parabolic and the annulus can be conformally parameterized by  $\mathbb{C}/\tau\mathbb{Z}$ . We will consider in the following conformal minimal periodic immersions  $X : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  with  $X(z + \tau) = X(z)$ .

Since  $dh \neq 0$ , the Hopf differential  $Q$  has no zeroes. If  $h^*$  is the harmonic conjugate of  $h$ , we can use the holomorphic map  $i(h + i h^*) : \mathbb{C}^2 \rightarrow \mathbb{C}$  to parameterize the annulus by the conformal parameter  $z = x + iy$ . In this parametrization the period of the annulus is  $\tau \in \mathbb{R}$  and

$$X(z) = (G(z), y) \text{ with } X(z + \tau) = X(z).$$

We say that we have parameterized the surface by its *third component*. We remark that  $Q = \frac{1}{4}(dz)^2$  and  $\omega$  satisfies the sinh-Gordon equation (1.1)

**Remark 3.1.** *In this paper we will relax the condition  $\tau \in \mathbb{R}$  into  $\tau \in \mathbb{C}$ , but we will parameterize our annuli conformally such that  $Q$  will be constant, independent of  $z$  and  $4|Q| = 1$ . We will say that the annulus is parameterized by its Sym point (see proposition A.1). This is a linear change in the conformal parameter  $z \rightarrow e^{i\Theta} z$*

In summary we have proven the following

**Theorem 3.2.** *A proper minimal annulus is parabolic and  $X : \mathbb{C}/\tau\mathbb{Z} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  has conformal parametrization  $X(z) = (G(z), h(z))$  with*

- (1) *constant Hopf differential  $Q = \frac{1}{4} \exp(i\Theta) dz^2$ , and  $h(z) = \operatorname{Re}(-ie^{i\Theta/2}z)$ .*
- (2) *The metric of the immersion is  $ds^2 = \cosh^2(\omega) dz \otimes d\bar{z}$ .*
- (3) *The third coordinate of the unit normal vector is  $n_3 = \tanh \omega$ .*
- (4) *The real function  $\omega : \mathbb{C}/\tau\mathbb{Z} \rightarrow \mathbb{R}$  is a solution of sinh-Gordon equation (1.1).*

**Annuli foliated by constant curvature curves.** The function  $\omega$  determines the geometry of the annulus. We are interested in a 2-parameter family of minimal annuli foliated by horizontal curves with constant geodesic curvature. In the case of minimal surfaces in  $\mathbb{R}^3$  Shiffman introduced the function  $u = -\lambda \partial_x(k_g)$ , called Shiffman's Jacobi field. In  $\mathbb{S}^2 \times \mathbb{R}$ , this function exists, and we cite the following

**Theorem 3.3.** [7] *Let  $A$  be a minimal surface embedded in  $\mathbb{S}^2 \times \mathbb{R}$ , transverse to every section of  $\mathbb{S}^2 \times \{t\}$  and parameterized by the third coordinate. Then the geodesic curvature in  $\mathbb{S}^2$  of the horizontal level curve  $\gamma_h(t) = A \cap (\mathbb{S}^2 \times \{t\})$  is given by*

$$(3.2) \quad k_g(\gamma_h) = \frac{-\omega_y}{\cosh \omega}.$$

*The function  $u = -\cosh \omega (k_g)_x$  is a Jacobi field, so that  $u$  is a solution of the elliptic equation*

$$\mathcal{L}u = \Delta_g u + \operatorname{Ric}(n)u + |dn|^2 u = 0.$$

*Here  $\operatorname{Ric}(n)$  is the Ricci curvature of the two planes tangent to  $A$ ,  $|dn|$  is the norm of the second fundamental form and  $\Delta_g = \frac{1}{\rho^2} \Delta$ .*

*Let  $A$  be a compact minimal annulus immersed in  $\mathbb{S}^2 \times \mathbb{R}$ , with  $\operatorname{Index}(L) \leq 1$ . If  $A$  is bounded by two curves  $\Gamma_1$  and  $\Gamma_2$  with constant geodesic curvature, then  $u$  is identically zero and  $A$  is foliated by horizontal curves of constant curvature in  $\mathbb{S}^2$ .*

This theorem proves the existence of a 2-parameter family of minimal surfaces foliated by horizontal constant geodesic curvature curves which are equivalent to Riemann's minimal example of  $\mathbb{R}^3$ . Meeks and Rosenberg [19] prove the existence by solving a Plateau problem between two geodesics  $\Gamma_1$  and  $\Gamma_2$  contained in two horizontal sections  $\mathbb{S}^2 \times \{t_1\}$  and  $\mathbb{S}^2 \times \{t_2\}$  and then find a stable minimal annulus bounded by the geodesics. By the theorem above this annulus is foliated by horizontal circles and using symmetries along horizontal geodesics in  $\mathbb{S}^2 \times \mathbb{R}$ , one obtains a properly embedded minimal annulus in  $\mathbb{S}^2 \times \mathbb{R}$ . This annulus is periodic in the third direction.

**Proposition 3.4.** [7] *A minimal annulus foliated by constant curvature horizontal curves admits a parametrization by the third coordinate where the metric  $ds^2 = \cosh(\omega) |dz|^2$  satisfies the Abresch system*

$$\begin{cases} \Delta \omega + \sinh \omega \cosh \omega = 0 \\ u = \cosh \omega \partial_x(k_g) = (\omega_{xy}) - \tanh \omega (\omega_x)(\omega_y) = 0. \end{cases}$$

Abresch [1] solved this system using elliptic functions and the separation of variable

$$\partial_x \frac{\omega_y}{\cosh \omega} = \partial_y \frac{\omega_x}{\cosh \omega} = \cosh^{-1} \omega (\omega_{xy} - \tanh \omega \omega_x \omega_y) = 0.$$

The solution  $\omega : \mathbb{C} \rightarrow \mathbb{R}$  allows to reconstruct the immersion up to the isometry (see section 6.3). The period closes in  $\mathbb{C}$  because horizontal curves are circles.

**Theorem 3.5.** [1], [7] *Let  $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a real-analytic solution of the Abresch system, then we define the functions  $f, g$  in terms of  $\omega$  by*

$$(3.3) \quad f = \frac{-\omega_x}{\cosh \omega} \quad \text{and} \quad g = \frac{-\omega_y}{\cosh \omega}$$



Then the real functions  $x \mapsto f(x)$  and  $y \mapsto g(y)$  depend on one variable and for  $c < 0, d < 0$  satisfy the system

$$\begin{aligned} -(f_x)^2 &= f^4 + (1 + c - d)f^2 + c, & -f_{xx} &= 2f^3 + (1 + c - d)f, \\ -(g_y)^2 &= g^4 + (1 + d - c)g^2 + d, & -g_{yy} &= 2g^3 + (1 + d - c)g. \end{aligned}$$

Conversely we can recover the solution  $\omega$  from functions  $f$  and  $g$  by

$$(3.4) \quad \sinh \omega = (1 + f^2 + g^2)^{-1}(f_x + g_y)$$

There is a solution of the system if and only if  $c \leq 0$  and  $d \leq 0$ ,  $\omega$  is doubly periodic and exists on the whole plane  $\mathbb{R}^2$ .

#### 4. CURVATURE ESTIMATE OF MEEKS AND ROSENBERG

Meeks and Rosenberg [18] study properly embedded minimal annuli in  $\mathbb{S}^2 \times \mathbb{R}$ . They prove an estimate of curvature in terms of the third coordinate of the flux.

**Lemma 4.1.** *Let  $\gamma$  be a simple curve (embedded) non homotopically trivial in  $A$  and let  $\eta = J\gamma'$  be a unit vector field tangent to  $A$  and orthogonal to  $\gamma'$  along  $\gamma$ , then we consider  $\eta_3 = \langle \eta, \frac{\partial}{\partial t} \rangle$  and the third coordinate of the flux map*

$$F_3 = \int_{\gamma} \eta_3 ds.$$

If the annulus  $A$  is conformally parameterized with  $Q = \frac{1}{4}e^{i\Theta}(dz)^2$  and  $X(z + \tau) = X(z)$  then  $F_3(\gamma) = \pm|\tau|$ .

*Proof.* After a conformal change of coordinate we assume that the annulus is parameterized by its third coordinate with real period  $|\tau|$ . We compute the flux along a horizontal curve  $x \rightarrow X(x, y_0)$ . The co-normal along this curve is  $\eta = \text{sech}(\omega)(G_y, 1)$ . Hence

$$F_3 = \int_0^{|\tau|} \eta_3 ds = |\tau|.$$

□

Meeks and Rosenberg proved in theorem 7.1 in [19] the following curvature estimate

**Theorem 4.2.** [19] *For any properly embedded minimal annulus  $A$  in  $\mathbb{S}^2 \times \mathbb{R}$  with flux  $|F_3| \geq \epsilon_0 > 0$ , there exists a constant  $C_1 > 0$  depending only  $\epsilon_0$  such that  $|K| \leq C_1(\epsilon_0)$ .*

They prove a linear growth estimate for minimal surfaces embedded in general product spaces  $M \times \mathbb{R}$ , but in  $\mathbb{S}^2 \times \mathbb{R}$  the annulus is parabolic and we can improve the result with a recent result of Mazet [15] and an estimate of Heintze-Karcher [10]. This implies that geometrically, an embedded annulus has an uniform tubular neighborhood.

**Lemma 4.3.** *If  $X : \mathbb{C}/\tau\mathbb{Z} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  is an embedded annulus, then  $X$  is the restriction of an  $\epsilon_1$ -tubular embedded neighborhood  $T_{\epsilon_1}$  of the annulus i.e. there is  $\epsilon_1 > 0$  such that*

$$Y : (\mathbb{C}/\tau\mathbb{Z}) \times ]-\epsilon_1, \epsilon_1[ \rightarrow \mathbb{S}^2 \times \mathbb{R} \text{ with } Y(z, s) = \text{Exp}_{X(z)}(s n(z))$$

*is an embedded three dimensional manifold  $T_{\epsilon_1} = Y((\mathbb{C}/\tau\mathbb{Z}) \times ]-\epsilon_1, \epsilon_1[)$  into  $\mathbb{S}^2 \times \mathbb{R}$ . This constant  $\epsilon_1$  depends only on a lower bound of the flux  $F_3 = |\tau| \geq \epsilon_0 > 0$ . Thus for any  $t > 0$ , there is a constant  $C_2 > 0$  which depends only on  $\epsilon_0$  such that*

$$\text{Area}(X \cap \mathbb{S}^2 \times [-t, t]) \leq C_2(\epsilon_0)t$$

*Proof.* We denote  $A(s) = Y(\mathbb{C}/\tau\mathbb{Z}, s)$ . Following [10] the differential of the exponential map  $Y : (z, s) \rightarrow \text{Exp}_{X(z)}(sn(p))$  is uniformly bounded on  $\mathbb{C} \times [-\epsilon_1, \epsilon_1]$  for  $\epsilon_1 > 0$  depending only on the geometry of  $\mathbb{S}^2 \times \mathbb{R}$ , and the upper bound of the Gaussian curvature  $K$  of  $A(0) = X(\mathbb{C}/\tau\mathbb{Z})$ . Then the projection  $\pi_s$  along geodesics of the equidistant surface  $A(s)$  to  $A(0)$  is a quasi-isometry. There is a constant  $K_1$  such that if  $|K| \leq C_1(\epsilon_0)$  and  $s \in [-\epsilon_1, \epsilon_1]$  we have  $K_1^{-1}(\epsilon_0)|v| \leq |d\pi_s(v)| \leq K_1(\epsilon_0)|v|$  for any  $v \in T_{Y(z,s)}A(s)$ .

Since the Ricci-curvature is positive, we have  $\frac{d}{ds}H_s = (\text{Ric}(\partial_s) + |dn_s|^2) \geq 0$  and the equidistant surface  $A(s_0)$  has mean curvature vector pointing outside the tubular neighborhood  $T_{s_0}$ .

Each equidistant surface  $A(s)$  has shape operator which satisfies a Riccati-type equation, hence by Karcher [13], the second fundamental form of  $A(s)$  is uniformly bounded on  $[-\epsilon_1, \epsilon_1]$ .

We satisfy the hypothesis of theorem 7 of Mazet [15]. If there is a parabolic annulus  $X$  such that  $T_{\epsilon_1} = Y((\mathbb{C}/\tau\mathbb{Z}) \times [\epsilon_1, \epsilon_1])$  is not embedded, a subregion of  $A$  would produce a connected component  $S$  bounded or unbounded into the tubular neighborhood  $T_{\epsilon_1}$ , which contradicts the maximum principle (see [15]).

This uniform bound of the minimal width of the tubular embedded neighborhood of the surface gives a linear growth estimate. There is a constant  $C$  depending only on the geometry of  $\mathbb{S}^2 \times \mathbb{R}$  and  $\epsilon_0$  such that

$$\epsilon_1 \text{Area}(X \cap \mathbb{S}^2 \times [-t, t]) \leq C \text{Vol}(T_{\epsilon_1} \cap [-t, t]) \leq 4C\pi t$$

and the constant  $C_2 = 4C\pi/\epsilon_1$  depends only on  $\epsilon_0$ .  $\square$

In the following, we will deform minimal annuli keeping  $F_3$  bounded away from zero. Then the curvature of the annulus will remain uniformly bounded. As a corollary we derive a uniform estimate for  $\omega$ , hence for the third coordinate of the normal  $n_3 = \tanh \omega$ .

**Proposition 4.4.** *If  $A$  is a properly embedded minimal annulus in  $\mathbb{S}^2 \times \mathbb{R}$  with induced metric  $ds^2 = \cosh^2 \omega |ds|^2$ , then there exist a constant  $C_0 > 0$  such that for any  $k \in \mathbb{N}$*

$$|\omega|_{A,k,\alpha} \leq C_0.$$

*Proof.* Consider a sequence of points  $p_n \in A$  such that  $\omega(p_n)$  is diverging to infinity, and consider a sequence of translations  $t_n e_3$  such that  $A + t_n e_3$  is a sequence of annuli with  $p_n + t_n e_3 \in \mathbb{S}^2 \times \{0\}$ . Then by the curvature estimate of Meeks-Rosenberg there is a subsequence converging locally to an embedded minimal surface  $A_0$  in  $\mathbb{S}^2 \times [-t, t]$ . The area estimate shows that  $A_0$  is an annulus, with the same flux  $F_3$ . By our hypothesis this leads to a pole occurring at the height  $t = 0$  since  $|\omega| \rightarrow \infty$ . The limit normal vector  $n_3(p_n) = \tanh \omega_n(p_n) \rightarrow \pm 1$  and the annulus  $A_0$  would be tangent to the height  $\mathbb{S}^2 \times \{0\}$ , a contradiction to the maximum principle. Then

$$\sup_{z \in A} |\omega| \leq C_0.$$

Now we apply the Schauder estimate to the sinh-Gordon equation to have a  $\mathcal{C}^{k,\alpha}$  estimate on the solution of the sinh-Gordon equation on  $\mathbb{R}/\tau \mathbb{Z} \times \mathbb{R}$ .  $\square$

Adapting an argument of Lockhart-McOwen [14], Meeks-Perez-Ros [17] prove the following:

**Theorem 4.5.** *An elliptic operator  $Lu = \Delta u + qu$  on a cylinder  $\mathbb{S}^1 \times \mathbb{R}$  has for bounded and continuous  $q$  a finite dimensional kernel on the space of uniformly bounded  $\mathcal{C}^2$  functions on  $\mathbb{S}^1 \times \mathbb{R}$ .*

## 5. FINITE TYPE THEORY OF THE SINH-GORDON EQUATION

**Pinkall-Sterling induction.** Suppose  $\omega$  is a solution of the sinh-Gordon equation (1.1). There is an iteration of Pinkall-Sterling [21] to obtain a hierarchy of solutions of the linearized sinh-Gordon equation (2.3). This inductively defines an infinite sequence of formal solutions of the linearized sinh-Gordon equation. To construct this family, we consider the matrix valued 1-form

$$(5.1) \quad \alpha(\omega) = \frac{1}{4} \begin{pmatrix} 2\omega_z & \mathbf{i} \lambda^{-1} e^\omega \\ \mathbf{i} \gamma e^{-\omega} & -2\omega_z \end{pmatrix} dz + \frac{1}{4} \begin{pmatrix} -2\omega_{\bar{z}} & \mathbf{i} \bar{\gamma} e^{-\omega} \\ \mathbf{i} \lambda e^\omega & 2\omega_{\bar{z}} \end{pmatrix} d\bar{z}$$

**Definition 5.1.** We call a solution of the sinh-Gordon equation  $\omega : \mathbb{C} \rightarrow \mathbb{R}$  of finite type, if there exists  $g \in \mathbb{N}$  and  $\gamma \in \mathbb{S}^1$  and functions  $u_n, \tau_n, \sigma_n : \mathbb{C} \rightarrow \mathbb{C}$  such that

$$(5.2) \quad \Phi_\lambda = \frac{\lambda^{-1}}{4} \begin{pmatrix} 0 & \mathbf{i} e^\omega \\ 0 & 0 \end{pmatrix} + \sum_{n=0}^g \lambda^n \begin{pmatrix} u_n(z) & e^\omega \tau_n(z) \\ e^\omega \sigma_n(z) & -u_n(z) \end{pmatrix}$$

is a solution of the Lax equation equation  $d\Phi_\lambda = [\Phi_\lambda, \alpha(\omega)]$ .

Pinkall and Sterling solve this problem and we can summarize their result in the following

**Proposition 5.2.** Suppose  $\Phi_\lambda$  is of the form (5.2) and  $\alpha(\omega)$  defined by (2.4) for a given real function  $\omega : \mathbb{C} \rightarrow \mathbb{R}$ . If  $\Phi_\lambda$  satisfies  $d\Phi_\lambda = [\Phi_\lambda, \alpha(\omega)]$  then:

- (1) The function  $\omega$  is a solution of the sinh-Gordon equation  $\Delta\omega + \cosh \omega \sinh \omega = 0$ .
- (2) For all  $n \in \mathbb{N}$ , the function  $u_n$  solves the linearized sinh-Gordon equation  $\Delta u_n + \cosh(2\omega) u_n = 0$ .
- (3) For given  $u_n, \sigma_n, \tau_{n-1}$ , with  $u_n$  solution of (2.3), we solve the system

$$\tau_{n;\bar{z}} = \frac{1}{2} \mathbf{i} \bar{\gamma} e^{-2\omega} u_n, \quad \tau_{n;z} = -2 \mathbf{i} \bar{\gamma} u_{n;zz} + 4 \mathbf{i} \bar{\gamma} \omega_z u_{n;z}$$

and then define

$$u_{n+1} = -2 \mathbf{i} \tau_{n;z} - 4 \mathbf{i} \omega_z \tau_n, \quad \sigma_{n+1} = \gamma e^{2\omega} \tau_n + 4 \mathbf{i} \gamma u_{n+1;\bar{z}}$$

to obtain  $u_{n+1}, \sigma_{n+1}, \tau_n$ . This defines a formal solution  $\Phi_\lambda$  of the Lax equation.

- (4) At each step,  $\tau_n$  is defined up to a complex constant  $c_n$  and
- (5) If we consider  $u_n = \bar{u}_{n+1}, \sigma_n = \bar{\tau}_n, \tau_{n-1} = \bar{\sigma}_{n+1}$ , then the iteration procedure of (3) gives  $u_{n+1} = \bar{u}_n, \sigma_{n+1} = \bar{\tau}_{n-1}, \tau_n = \bar{\sigma}_n$  and

$$\tilde{\Phi}_\lambda(z) := (\lambda^{g-1})^t \overline{\Phi_{1/\bar{\lambda}}(z)}$$

is also a solution of  $d\Phi_\lambda = [\Phi_\lambda, \alpha(\omega)]$ .

Pinkall-Sterling [21] prove that beginning the iteration with  $u_{-1} = 0, \sigma_{-1} = 0, \tau_{-1} = 1/4$  and  $u_0 = \omega_z$ , then all solutions depend only on  $\omega$  and its  $k$ -th derivatives with  $k \leq 2n + 1$  (see Proposition 3.1 in [21]). Each function constructed in this way is complex valued, and  $u_n : \mathbb{C} \rightarrow \mathbb{C}$  has real and imaginary part that are both solutions of (2.3). Applying this iteration procedure with  $u_{-1} = 0$ , we obtain the sequence with first terms

$$u_{-1} = 0, \quad u_0 = \omega_z, \quad u_1 = (\omega_{zzz} - 2\omega_z^3) \\ u_2 = (\omega_{zzzzz} - 10\omega_{zzz}\omega_z^3 - 10\omega_{zz}^2\omega_z + 6\omega_z^5), \dots$$

Since we consider a uniform bounded solution of the sinh-Gordon equation  $\omega : \mathbb{R} \times \mathbb{S}^1 \rightarrow \mathbb{R}$ , this infinite sequence produces bounded Jacobi fields on  $A$  by Schauder estimates. Hence this sequence is a finite dimensional family (by theorem 4.5) and there is a  $g \in \mathbb{N}$  and complex coefficient  $a_i, b_i$  such that

$$\sum_{i=0}^g a_i u_i + b_i \bar{u}_i = 0.$$

By prescribing the right constants  $c_0, c_1, \dots$  in the iteration procedure the  $g + 1$  term is zero. Thus the relation above implies the existence of a polynomial solution  $\Phi_\lambda$  of degree  $g$ , hence  $\omega$  is of finite type.

**Theorem 5.3.** *A properly embedded minimal annulus in  $\mathbb{S}^2 \times \mathbb{R}$  is of finite type*

*Proof.* Due to the Meeks-Rosenberg theorem, the curvature of this annulus is bounded by (2.1) where  $F_3$  is the third coordinate of the flux. From this estimate and a slide-back sequence we prove (see Lemma 4.4) that the third coordinate defined by  $n_3 = \tanh \omega$  is bounded away from zero and  $\sup_A |\omega| \leq C_0$ .

Since  $n_3 \neq 1$ , the tangent plane is nowhere horizontal and  $(h_z)^2 = -Q \neq 0$  on the annulus. By Theorem 3.2 the properly embedded annulus is parabolic and we can re-parameterize by its third coordinate. We consider an immersion  $X : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  with  $X(x + iy) = (G(x + iy), y)$  and period  $\tau \in \mathbb{R}$  defined by the smallest positive value such that  $X(z + \tau) = X(z)$ . The metric is given by  $ds^2 = \cosh^2 \omega |dz|^2$  and  $n_3 = \tanh \omega$  is the third coordinate of the normal. The function  $\omega : \mathbb{C} \rightarrow \mathbb{R}$  is uniformly bounded and satisfies the sinh-Gordon equation (1.1). Schauder estimates apply and we have

$$|\omega|_{\mathcal{C}^{k,\alpha}} \leq C_1$$

for a constant  $C_1$  depending on the annulus  $A = X(\mathbb{C}/\tau\mathbb{Z})$ . Now we apply the theorem of Mazzeo-Pacard which assures that the operator

$$\Delta + \cosh(2\omega) : \mathcal{C}^{2,\alpha}(\mathbb{C}/\tau\mathbb{Z}) \rightarrow \mathcal{C}^{0,\alpha}(\mathbb{C}/\tau\mathbb{Z})$$

has bounded dimensional kernel on the subspace of uniformly bounded functions on the annulus  $A$ . Hence  $\omega$  is of finite type by the Pinkall-Sterling iteration.  $\square$

**Polynomial Killing field and potential.** For  $\Phi_\lambda$  in (5.2) we choose the constant  $\gamma \in \mathbb{S}^1$  in  $\alpha_\lambda$  (see formula 2.4) such that the residue at  $\lambda = 0$  of

$$\zeta_\lambda(z) = \Phi_\lambda(z) - \lambda^{g-1} \overline{\Phi_{1/\bar{\lambda}}(z)}^t$$

is of the form  $\begin{pmatrix} 0 & i\mathbb{R}^\times \\ 0 & 0 \end{pmatrix}$ , so that  $\zeta_\lambda$  takes values in the space of potentials  $\mathcal{P}_g$  (see Definition 2.1).

**Definition 5.4.** *Polynomial Killing fields are maps  $\zeta_\lambda : \mathbb{C} \rightarrow \mathcal{P}_g$  which solve*

$$d\zeta_\lambda = [\zeta_\lambda, \alpha(\omega)] \text{ with } \zeta_\lambda(0) = \xi_\lambda \in \mathcal{P}_g.$$

When  $g$  is even,  $\xi_0, \dots, \xi_{\frac{g}{2}-1}$  are independent  $2 \times 2$  traceless complex matrix and the real vector space  $\mathcal{P}_g$  is of dimension  $3g + 1$ , and has up to isomorphism a unique norm  $\|\cdot\|$ . For an odd  $g$ , the difference is in  $\hat{\xi}_{\frac{g-1}{2}} \in \mathfrak{su}_2(\mathbb{C})$ , the Lie algebra of  $\mathfrak{su}_2(\mathbb{C})$ . These Laurent polynomials  $\xi_\lambda$  define smooth mappings  $\xi : \lambda \in \mathbb{S}^1 \rightarrow \mathfrak{sl}_2(\mathbb{C})$ .

**Remark 5.5.** *The polynomial  $a(\lambda) := -\lambda \det \xi_\lambda$  satisfies the reality condition*

$$(5.3) \quad \lambda^{2g} \overline{a(1/\bar{\lambda})} = a(\lambda).$$

Since  $\chi_\lambda = \lambda^{\frac{1-g}{2}} \xi_\lambda$  is traceless and satisfy  ${}^t \overline{\chi_{1/\bar{\lambda}}} = -\chi_\lambda$  for any  $\xi_\lambda \in \mathcal{P}_g$  and for  $\lambda \in \mathbb{S}^1$ , the determinant is the square of a norm and we have  $\|\xi_\lambda\| = \|\chi_\lambda\| = \det \chi_\lambda \geq 0$  for  $\lambda \in \mathbb{S}^1$ . Thus

$$(5.4) \quad \lambda^{-g} a(\lambda) \leq 0 \text{ for } \lambda \in \mathbb{S}^1$$

The condition  $\text{trace}(\hat{\xi}_{-1} \hat{\xi}_0)$  implies that  $a(0) \neq 0$  and by symmetry the highest coefficient of  $a$  is non-zero (see definition 6.6 to see that  $\text{trace}(\hat{\xi}_{-1} \hat{\xi}_0) = \beta_{-1} \gamma_0 = a(0)$ ).

**Spectral curve.** The spectral curve is defined by the determinant of a polynomial Killing field  $\zeta_\lambda$ . The main property of the Lax equation is that  $a(\lambda) = -\lambda \det \zeta_\lambda = -\lambda \det \xi_\lambda$  is independent of  $z$ . Following Bobenko ([2]), the polynomial  $a(\lambda)$  defines a hyperelliptic Riemann surface  $\Sigma$  as follows:

**Definition 5.6.** *The spectral curve  $\Sigma$  of genus  $g$  associate to a potential  $\xi_\lambda \in \mathcal{P}_g$  is defined by adding  $(\infty, 0)$  and  $(\infty, \infty)$  as branch points in the compactification of (2.5)*

$$\Sigma^* = \{(\nu, \lambda) \in \mathbb{C}^2 \mid \det(\nu \mathbb{Id} - \zeta_\lambda) = 0\} = \{(\nu, \lambda) \in \mathbb{C}^2 \mid \nu^2 = \lambda^{-1} a(\lambda) = -\det \xi_\lambda\}$$

where  $a$  is a polynomial with  $2g$  pairwise distinct roots which satisfies the reality conditions

$$|a(0)| = \frac{1}{16}, \quad \lambda^{2g} \overline{a(\bar{\lambda}^{-1})} = a(\lambda) \quad \text{and} \quad \frac{a(\lambda)}{\lambda^g} \leq 0 \quad \text{for all } \lambda \in \mathbb{S}^1.$$

**Proposition 5.7.**  $\Sigma$  has three involutions

$$(5.5) \quad \sigma : (\lambda, \nu) \mapsto (\lambda, -\nu) \quad \rho : (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, -\bar{\lambda}^{1-g} \bar{\nu}) \quad \eta : (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, \bar{\lambda}^{1-g} \bar{\nu})$$

The involution  $\sigma$  is the hyperelliptic involution. The involution  $\rho$  has no fixed point while  $\eta$  fixes all points of the unit circle  $|\lambda| = 1$ . In particular, roots  $\alpha_1, \dots, \alpha_{2g}$  of  $a(\lambda)$  are symmetric with respect to the unit circle so that  $a(\alpha_i) = 0$  if and only if  $a(1/\bar{\alpha}_i) = 0$ .

## 6. CONSTRUCTION OF MINIMAL ANNULI VIA POTENTIALS

We next explain how to reconstruct an immersion from a potential  $\xi_\lambda \in \mathcal{P}_g$ . Expanding a polynomial Killing field  $\zeta_\lambda : \mathbb{C} \rightarrow \mathcal{P}_g$  as

$$(6.1) \quad \zeta_\lambda = \begin{pmatrix} 0 & \beta_{-1} \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & -\alpha_0 \end{pmatrix} \lambda^0 + \dots + \begin{pmatrix} \alpha_g & \beta_g \\ \gamma_g & -\alpha_g \end{pmatrix} \lambda^g$$

we associate a matrix 1-form defined by

$$(6.2) \quad \alpha(\zeta_\lambda) = \begin{pmatrix} \alpha_0 & \beta_{-1} \lambda^{-1} \\ \gamma_0 & -\alpha_0 \end{pmatrix} dz - \begin{pmatrix} \bar{\alpha}_0 & \bar{\gamma}_0 \\ \bar{\beta}_{-1} \lambda & -\bar{\alpha}_0 \end{pmatrix} d\bar{z}$$

We cite Proposition 3.2 [8] which provides the following existence and uniqueness result

**Proposition 6.1.** [8] *For each  $\xi_\lambda \in \mathcal{P}_g$  there is a unique solution  $\zeta_\lambda(z) : \mathbb{C} \rightarrow \mathcal{P}_g$  of*

$$(6.3) \quad d\zeta_\lambda(z) = [\zeta_\lambda(z), \alpha(\zeta_\lambda(z))] \quad \text{with} \quad \zeta_\lambda(0) = \xi_\lambda.$$

If we set  $4\beta_{-1}(z) := ie^{\omega(z)}$  then  $\omega$  is a solution of sinh-Gordon equation and  $\alpha(\zeta_\lambda(z)) = \alpha(z)$  has the form of formula (2.4) and we express the coefficient of  $\alpha(z)$  in terms of  $\omega$ .

**Remark 6.2.** *The Lax equation preserves  $\beta_{-1}(z) \in i\mathbb{R}^+$  and we can define a function  $\omega : \mathbb{C} \rightarrow \mathbb{R}$  by setting  $4\beta_{-1}(z) := ie^{\omega}$ . Now the equation  $\beta_{-1,z} = 2\alpha_0\beta_1$  implies that  $2\alpha_0 = \omega_z$ . To express  $\gamma_0$  in terms of  $\alpha_0$  and  $\beta_{-1}$ , we consider the Lax equation and find  $d''\gamma_0 = -2\bar{\alpha}_0\gamma_0$ . Then  $\gamma_0 = qe^{-\omega}$  where  $q$  is a holomorphic function. The term  $q$  is constant. The reason is that along the parameter  $z$ , we have  $a(\lambda) = -\lambda \det \zeta_\lambda(z) = -\lambda \det \xi_\lambda$  and  $a(0) = \beta_{-1}\gamma_0 = q/4$  with  $4|q| = 1$ . The polynomial coefficient of  $\zeta_\lambda(z)$  depends on higher derivative of the function  $\omega(z)$  pointwise in  $z$ .*

From the potential we obtain the extended frame and then the immersion by Sym-Bobenko formula

**Theorem 6.3.** [8] *Let  $\xi_\lambda$  be a potential and  $\zeta_\lambda : \mathbb{C} \rightarrow \mathcal{P}_g$  the polynomial Killing field (6.3). For any constant  $\gamma$  in  $\mathbb{S}^1$ , the immersion*

$$X_\lambda(z) = (F_\lambda \sigma_3 F_\lambda^{-1}, \text{Re}(-i\sqrt{\gamma\lambda^{-1}}z))$$

with  $\lambda \in \mathbb{S}^1$ , defines a one-parameter family of conformal minimal immersion in  $\mathbb{S}^2 \times \mathbb{R}$  with metric  $ds^2 = \cosh^2 \omega |dz|^2$  if the extended frame  $F_\lambda : \Omega \rightarrow \mathrm{SL}_2(\mathbb{C})$  is a solution of

$$(6.4) \quad F_\lambda^{-1} dF_\lambda = \alpha(\zeta_\lambda(z)) := \alpha(\omega) \quad \text{with} \quad F_\lambda(0) = \mathbb{I}d$$

where  $\alpha(\omega)$  has the form of (2.4). For  $\lambda \in \mathbb{S}^1$  we obtain the one parameter family of isometric associated family. In particular for  $\lambda = \gamma = 1$  we have  $X_1 = (G_1, y)$  a conformal immersion parameterized by its third coordinate. The function  $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$  solves the sinh-Gordon equation.

**Remark 6.4. Reality condition.** For  $\lambda \in \mathbb{S}^1$ ,  $\alpha(\omega)$  takes values in  $\mathfrak{su}_2$  and then  $F_\lambda$  takes values in  $\mathrm{SU}_2$ . For general  $\lambda \in \mathbb{C}^*$  we look for solution  $F_\lambda \in \mathrm{SL}_2(\mathbb{C})$ . We call  $\lambda$  the spectral parameter. From the relation  $\bar{\alpha}_{1/\bar{\lambda}}^t = -\alpha_\lambda$ , the solution of (6.4) satisfies  $\bar{F}_{1/\bar{\lambda}}^t = F_\lambda^{-1}$ .

**Remark 6.5. Normalisation.** By conformal parametrization we can choose,  $4|Q| = 1$  (the annulus is transverse to horizontal sections). Next we discuss the constants  $\beta, \gamma \in \mathbb{S}^1$  which are related to the Hopf differential. We will see that  $4Q = \beta\gamma\lambda^{-1}$ . We can normalize the parametrization by  $\beta = \mathbf{i}$  and a constant  $|\gamma| = 1$ . For a given extended frame  $F_\lambda$  which satisfies the equation (6.4), we consider  $g = \begin{pmatrix} \beta & 0 \\ 0 & \bar{\beta} \end{pmatrix} \in U(1)$ . Then  $\tilde{F}_\lambda = F_\lambda g$  induces the same immersion  $G_\lambda = F_\lambda \sigma_3 F_\lambda^{-1}$  and satisfies equation (6.4) with  $\beta = 1$ ,  $\tilde{F}_\lambda^{-1} d\tilde{F}_\lambda = g^{-1} \alpha_\lambda(z) g$ .

**The Sym point and conformal parametrization.** The determinant of  $\zeta_\lambda(z)$  does not depend on  $z$  and we remark that  $k(\lambda) = \lambda \det \zeta_\lambda(z) = \lambda \det \xi_\lambda$  and  $k(0) = -\beta_{-1}\gamma_0 = -a(0)$ . The term  $\beta_{-1}\gamma_0$  of the corresponding polynomial Killing field does not depend on the surface parameter  $z$ .

If  $F_\lambda$  is related to a minimal surface  $X_\lambda : \mathbb{R}^2 \rightarrow \mathbb{S}^2 \times \mathbb{R}$  parameterized by its third coordinate then there is  $\lambda_0 \in \mathbb{S}^1$ , such that

$$X_{\lambda_0}(z) = (G_{\lambda_0}(z), y) = (F_{\lambda_0}(z) \sigma_3 F_{\lambda_0}^{-1}(z), \mathrm{Re}(-\mathbf{i} \sqrt{\beta_{-1}\gamma_0 \lambda_0^{-1}} z))$$

and we observe that then the Hopf differential is  $Q = -4\beta_{-1}\gamma_0 \lambda_0^{-1} (dz)^2 = \frac{1}{4} (dz)^2$  i.e.  $\beta_{-1}\gamma_0 = -\frac{\lambda_0}{16}$ . The value of  $\lambda_0 = e^{i\theta}$  associate to an immersion  $X_\lambda$  is called the Sym point. In the following we will prefer a conformal parametrization which fixes the Sym point to  $\lambda_0 = 1$ . To do that we make the conformal change  $z \rightarrow e^{i(1-g)\theta/2} z$  and the Möbius transformation  $\lambda \rightarrow e^{i\theta} \lambda$ . We explain in Appendix A how to apply this transformation.

**Definition 6.6.** A finite type minimal immersion  $X : \mathbb{R}^2 \rightarrow \mathbb{S}^2 \times \mathbb{R}$  is conformally parameterized by its Sym point if there is a polynomial Killing field

$$\zeta_\lambda : \mathbb{R}^2 \rightarrow \left\{ \xi_\lambda \in \mathcal{P}_g \mid \lambda \det \xi_\lambda = -a(\lambda) \text{ and } \beta_{-1}\gamma_0 = a(0) = -\frac{1}{16} e^{i(1-g)\theta} := -\frac{1}{16} e^{i\Theta} \right\}$$

which solves the Lax equation (6.3) where  $a(\lambda)$  is a complex polynomial of degree  $2g$  which satisfies the reality conditions (2.6) (see remark 6.4) If  $F_\lambda$  is the unitary factor associate to  $\xi_\lambda$ , then in this parametrization the immersion is given by (2.7)

## 7. ISOSPECTRAL GROUP AND SPECTRAL DATA

In this Section we characterize those potentials  $\xi_\lambda \in \mathcal{P}_g$ , which correspond to periodic minimal immersions. This property turns out to be a property of the spectral curve in definition 5.6. For a given polynomial  $a$  of degree  $2g$  obeying (2.6) either all elements of  $I(a)$  in (2.9) have this property or no element. These sets  $I(a)$  are called isospectral sets and consists of matrices  $\xi_\lambda$  having the same spectral curves  $\Sigma$  and same off-diagonal product  $a(0) = \beta_{-1}\gamma_0$ . It is a  $g$ -dimensional complex manifold (see below) and its tangent space at an initial value is associate to the solution  $u_1, u_2, \dots$  of LSG depending on  $\omega$  and its higher derivative. Each solution of LSG constructed in proposition 5 integrates to a long time solution  $\omega(t)$  of the sinh-Gordon equation. This deformation gives a deformation of annuli preserving the spectral curve. The integrable

system consists of this hierarchy of commuting variational fields. These variational fields are exactly the isospectral deformations, and the commuting property is the key point to describe the geometry of the annulus at infinity, as well as the embeddedness property.

**Definition 7.1.** Suppose  $X : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  is a minimal immersion of finite type with potential  $\xi$ . An isospectral deformation of  $X$  is a smooth family of finite type immersions  $X(t, z) : [0, T] \times \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  with  $X(0, z) = X(z)$ , and corresponding smooth family of potentials  $\xi_\lambda : [0, T] \rightarrow I(a)$  with  $a(\lambda) = -\lambda \det \xi_\lambda(0) = -\lambda \det \xi_\lambda(t)$  for all  $t \in [0, T]$ .

**Iwasawa decomposition.** For real  $r \in (0, 1]$ , let  $\mathbb{S}_r = \{\lambda \in \mathbb{C}; |\lambda| = r\}$  and define the loop group  $\Lambda_r \text{SL}_2(\mathbb{C}) = \mathcal{O}(\mathbb{S}_r, \text{SL}_2(\mathbb{C}))$  as the set of analytic maps  $\mathbb{S}_r \rightarrow \text{SL}_2(\mathbb{C})$ . We use the following loop group and subgroup: Let the circle  $\mathbb{S}_r = \{\lambda \in \mathbb{C}; |\lambda| = r\}$ , the disc  $I_r = \{\lambda \in \mathbb{C}; |\lambda| < r\}$  and the annulus  $A_r = \{\lambda \in \mathbb{C}; r < |\lambda| < 1/r\}$ . We consider the set of analytic maps

$$\begin{aligned} \Lambda_r \text{SL}_2(\mathbb{C}) &= \{F : \mathbb{S}_r \rightarrow \text{SL}_2(\mathbb{C}) \mid F \text{ analytic}\}, \\ \Lambda_r \text{SU}_2(\mathbb{C}) &= \{F : A_r \rightarrow \text{SL}_2(\mathbb{C}) \mid F \text{ analytic on } A_r \text{ and } F|_{\mathbb{S}^1} \in \text{SU}_2\}, \\ \Lambda_r^+ \text{SL}_2(\mathbb{C}) &= \{B \in \Lambda_r \text{SL}_2(\mathbb{C}) \cap \mathcal{O}(I_r, \text{SL}_2(\mathbb{C})) \mid B(0) = \begin{pmatrix} \rho & c \\ 0 & 1/\rho \end{pmatrix} \text{ for } \rho \in \mathbb{R}^+ \text{ and } c \in \mathbb{C}\}. \end{aligned}$$

Then multiplication is a real analytic diffeomorphism

$$\Lambda_r \text{SL}_2(\mathbb{C}) \rightarrow \Lambda_r \text{SU}_2(\mathbb{C}) \times \Lambda_r^+ \text{SL}_2(\mathbb{C})$$

and any  $\phi(\lambda) \in \Lambda_r \text{SL}_2(\mathbb{C})$  can be uniquely factorized into  $\phi(\lambda) = F_\lambda B_\lambda$  with  $F_\lambda \in \Lambda_r \text{SU}_2(\mathbb{C})$  and  $B_\lambda \in \Lambda_r^+ \text{SL}_2(\mathbb{C})$ . This is the  $r$ -Iwasawa decomposition (or factorization) of  $\phi(\lambda)$ . When  $r = 1$ , we omit the subscript and it is referred to as the Iwasawa decomposition. Given a map  $\phi_\lambda : \Omega \rightarrow \Lambda_r \text{SL}_2(\mathbb{C})$ , we can apply the decomposition pointwise on the domain  $\Omega$ , and then  $F : \mathbb{C} \rightarrow \Lambda_r \text{SU}_2(\mathbb{C})$  and  $B : \mathbb{C} \rightarrow \Lambda_r^+ \text{SL}_2(\mathbb{C})$ .

**Theorem 7.2.** [16, 22] The multiplication  $\Lambda_r \text{SU}_2(\mathbb{C}) \times \Lambda_r^+ \text{SL}_2(\mathbb{C}) \rightarrow \Lambda_r \text{SL}_2(\mathbb{C})$  is a real analytic bijective diffeomorphism. The unique splitting of an element  $\phi(\lambda) \in \Lambda_r \text{SL}_2(\mathbb{C})$  into

$$\phi(\lambda) = F_\lambda B_\lambda$$

with  $F_\lambda \in \Lambda_r \text{SU}_2(\mathbb{C})$  and  $B_\lambda \in \Lambda_r^+ \text{SL}_2(\mathbb{C})$  is the  $r$ -Iwasawa decomposition.

**Isospectral Group action.** We define a smooth action of the Lie-group  $\mathbb{C}^g$  on  $I(a)$ . This action corresponds to a deformation by integrating the hierarchy of solutions of LSG of Proposition 5.2.

**Definition 7.3.** For  $t = (t_0, \dots, t_{g-1}) \in \mathbb{C}^g$  we set the group action  $\pi(t) : I(a) \rightarrow I(a)$  defined by

$$\begin{aligned} \xi_\lambda &\mapsto \pi(t)\xi_\lambda = B_\lambda(t)\xi_\lambda B_\lambda^{-1}(t) = F_\lambda^{-1}(t)\xi_\lambda F_\lambda(t) \\ (7.1) \quad &\text{with } \exp\left(\xi_\lambda \sum_{i=0}^{g-1} \lambda^{-i} t_i\right) = F_\lambda(t) B_\lambda(t). \end{aligned}$$

In the last equation the right hand side is the Iwasawa decomposition of the left hand side.

**Remark 7.4.** i) The first solution of LSG in the hierarchy is  $u_0 = \omega_z$ , which acts transitively on the annulus. The determinant  $\det \xi_\lambda$  is invariant under this action and the action of the translations by  $z \in \mathbb{C}$  coincides with the action of  $t_z = (z, 0, \dots, 0) \in \mathbb{C}^g$ . The associate Jacobi field is obtained by changing coordinates and by vertical translations.

The decomposition  $\exp(z\xi_\lambda) = F_\lambda(z)B_\lambda(z)$  gives the polynomial Killing field  $\zeta_\lambda(z) = F_\lambda^{-1}(z)\xi_\lambda F_\lambda(z)$ .

ii) The second solution of LSG in the hierarchy is  $u_1 = \omega_{zzz} - 2\omega_z^3$  which is associated to the Schiffmann Jacobi field on the surface.

For  $t, t' \in \mathbb{C}^g$  the corresponding Iwasawa decompositions obey

$$F_\lambda(t+t')B_\lambda(t+t') = F_\lambda(t)B_\lambda(t)F_\lambda(t')B_\lambda(t') = F_\lambda(t')B_\lambda(t')F_\lambda(t)B_\lambda(t).$$

Therefore the Iwasawa decomposition for the action of  $\pi(t')$  on  $\pi(t)\xi$  is equal to

$$F'_\lambda(t')B'_\lambda(t') = \exp \left( B_\lambda(t)\xi_\lambda B_\lambda^{-1}(t) \sum_{i=0}^{g-1} \lambda^{-i} t_i \right)$$

and

$$F'_\lambda(t')B'_\lambda(t') = B_\lambda(t)F_\lambda(t')B_\lambda(t')B_\lambda^{-1}(t) = F_\lambda^{-1}(t)F_\lambda(t+t')B_\lambda(t+t')B_\lambda^{-1}(t).$$

Since the Iwasawa decomposition is unique we conclude

**Proposition 7.5.** *The group action  $\pi(t) : I(a) \rightarrow I(a)$  is a commuting action and*

$$\pi(t')\pi(t)\xi_\lambda = B'(t')B(t)\xi_\lambda B^{-1}(t)B'^{-1}(t') = B(t+t')\xi_\lambda B^{-1}(t+t') = \pi(t+t')\xi_\lambda.$$

$$F'_\lambda(t') = F_\lambda^{-1}(t)F_\lambda(t+t') \text{ and } B'_\lambda(t') = B_\lambda(t+t')B_\lambda^{-1}(t)$$

This action has an obvious extension to an action of all sequences  $(\dots, t_{-1}, t_0, t_1, \dots)$  with only finitely many non vanishing entries. Whenever the exponential on the left hand side of (7.1) belongs to one factor of the Iwasawa decomposition, then the corresponding  $t$  acts trivially. For example, for any  $k \geq 1$ , the matrix  $\exp(\lambda^k \xi_\lambda)$  belongs to the second factor of the Lie Algebra, while if  $k \leq -g$ , the matrix  $\exp(\lambda^k \xi_\lambda)$  is a first factor of the Lie Algebra of the Iwasawa decomposition. These actions acts trivially.

Furthermore, since all  $\xi_\lambda \in I(a)$  obey

$${}^t \overline{\xi_{1/\lambda}} = -\lambda^{1-g} \xi_\lambda$$

the matrices

$$\exp \left( \xi_\lambda \left( \sum_{i=0}^{g-1} \lambda^{-i} t_i + \sum_{i=0}^{g-1} \lambda^{i+1-g} \bar{t}_i \right) \right) \in \Lambda \text{SU}_2(\mathbb{C})$$

belong to the Lie Algebra of the first factor in the Iwasawa decomposition. Then the related action  $\pi(t_i + \bar{t}_{g-1+i})\xi_\lambda = \xi_\lambda$  is trivial. In summary we conclude with the following

**Remark 7.6.** *Only a finite dimensional Lie group with Lie algebra isomorphic to  $\mathbb{R}^g$  acts non-trivially on  $I(a)$ .*

This group action yields a differentiable structure on the isospectral set, and we cite

**Proposition 7.7.** [8] *Let  $I(a)$  be the isospectral set with polynomial  $a(\lambda)$ , which satisfies the reality condition of definition 6.6*

- (1)  *$I(a)$  is compact.*
- (2) *If the  $2g$  roots of  $a(\lambda)$  are pairwise distinct (without double roots), then  $I(a)$  is connected smooth  $g$ -dimensional manifold. This manifold is diffeomorphic to a  $g$ -dimensional real torus;  $I(a) \cong (\mathbb{S}^1)^g$ .*

Proposition 7.7 implies the following corollary:

**Corollary 7.8.** *We consider a finite type annulus  $X : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  with period  $\tau \in \mathbb{C}$  associate to a spectral curve  $\Sigma$  and potential  $\xi_\lambda$ .*

- (1) *If  $a(\lambda)$  has only simple roots, then the solution  $\omega$  is the restriction of a function on a  $g$ -dimensional torus to a two dimensional subgroup, and so  $\omega$  and  $\zeta_\lambda(z)$  are quasi-periodic.*
- (2) *Every solution  $u_n$  of LSG induced by the Pinkall-Sterling iteration integrates into a long time solution of the sinh-Gordon equation.*



- (3) If  $a(\lambda)$  has only simple roots, then every annulus has a quasi-periodic metric. This means that if we consider a diverging sequence  $(t_n)$ , the sequence annuli  $\omega(x, y + t_n) = \omega_n(x, y)$  has a converging subsequence  $\tilde{\omega}(x, y)$  defined by a potential in  $I(a)$ .

**Spectral data of minimal annuli.** We study the monodromy  $M_\lambda(\tau) = F_\lambda(z)^{-1}F_\lambda(z + \tau)$  of the extended frame  $F_\lambda$  for a period  $\tau$ . By construction the monodromy takes values in  $SU_2$  for  $|\lambda| = 1$ . The monodromy depends on the choice of base point  $z$ , but its conjugacy class and hence eigenvalues  $\mu, \mu^{-1}$  do not. The eigenspace of  $M_\lambda(\tau)$  depends holomorphically on  $(\mu, \lambda)$ .

Let  $\zeta_\lambda$  be a solution of Lax equation (6.1) with initial value  $\xi_\lambda \in \mathcal{P}_g$ , with period  $\tau$  so that  $\zeta_\lambda(z + \tau) = \zeta_\lambda(z)$  for all  $z \in \mathbb{R}^2$ . Then for  $z = 0$  we have

$$\xi_\lambda = \zeta_\lambda(0) = \zeta_\lambda(\tau) = F_\lambda^{-1}(\tau) \xi F_\lambda(\tau) = M_\lambda^{-1}(\tau) \xi M_\lambda(\tau)$$

and thus

$$[M_\lambda(\tau), \xi_\lambda] = 0.$$

Hence the eigenvalues of  $\xi_\lambda$  and  $M_\lambda(\tau)$  are different functions on the same Riemann surface  $\Sigma$ . Furthermore the eigenspaces of  $M_\lambda(\tau)$  and  $\xi_\lambda$  coincide point-wise. At  $\lambda = 0$  and  $\lambda = \infty$ , the monodromy  $M_\lambda(\tau) = F_\lambda(\tau)$  has essential singularities. The period  $\tau$  is related to a trivial action on the isospectral set  $I(a)$ , and we prove that the essential singularities of  $\mu$  at  $\lambda = 0$  and  $\lambda = \infty$  depend only on an isospectral orbit.

**Proposition 7.9.** [8] *The group  $\Gamma_{\xi_\lambda} = \{t \in \mathbb{R}^g; \pi(t)\xi_\lambda = \xi_\lambda\}$  depends only on the orbit of  $\xi_\lambda$ . If  $\gamma \in \Gamma_{\xi_\lambda}$  satisfies  $F(\gamma) = \pm \text{Id}$  at  $\lambda = \lambda_0$  for some  $\xi_{\lambda,0} \in \mathcal{P}_g$  then the same is true for all  $\xi_\lambda$  in the orbit of  $\xi_{\lambda,0}$ . The period  $\tau$  is related to  $t = (\tau, 0, \dots, 0) \in \Gamma_{\xi_\lambda}$ .*

*Proof.* This is the direct consequence of the commuting property of the group action  $\pi(t)$ . We have only to remark, that  $\pi(\gamma)\xi_\lambda = \xi_\lambda$  implies that  $\pi(\gamma)\pi(t)\xi_\lambda = \pi(t)\pi(\gamma)\xi_\lambda = \pi(t)\xi_\lambda$ .  $\square$

**Proposition 7.10.** [8] *Let  $\xi_\lambda \in \mathcal{P}_g$  without roots. Then  $\gamma \in \Gamma_{\xi_\lambda} = \{t \in \mathbb{R}^g \mid \pi(t)\xi_\lambda = \xi_\lambda\}$  if and only if there exists on  $\Sigma = \{(\nu, \lambda) \in \mathbb{C}^2 \mid \nu^2 = a(\lambda)/\lambda\}$  a function  $\mu$  which satisfies:*

- (1)  $\mu$  is holomorphic on  $\Sigma - \{0, \infty\}$  and there exist holomorphic functions  $f, g$  defined on  $\mathbb{C}^*$  with  $\mu = f\nu + g$ .
- (2)  $\sigma^*\mu = \mu^{-1}, \rho^*\mu = \bar{\mu}^{-1}, \eta^*\mu = \bar{\mu}$  and  $\mu = \pm 1$  at each branch point  $\alpha_i$  of  $\Sigma$ .
- (3)  $d \ln \mu$  is a meromorphic 1-form with  $d \ln \mu - d(\sum_{i=0}^{g-1} \gamma_i \lambda^{-i} \nu)$  holomorphic in a neighborhood of  $\lambda = 0$  and  $d \ln \mu + d(\sum_{i=0}^{g-1} \bar{\gamma}_i \lambda^{i+1-g} \nu)$  is holomorphic at  $\lambda = \infty$ .
- (4) For  $\gamma = (\tau, 0, \dots, 0)$ , the differential  $d \ln \mu$  has second order poles without residues at the two points  $\lambda = 0$  and  $\lambda = \infty$ , and  $d \ln \mu - \frac{ie^{i\Theta/2}\tau}{4} d\sqrt{\lambda}^{-1}$  extends holomorphically to  $\lambda = 0$ , while  $d \ln \mu - \frac{ie^{-i\Theta/2}\bar{\tau}}{4} d\sqrt{\lambda}$  extends holomorphically to  $\lambda = \infty$ .

The existence of an annulus depends on the existence of the function  $\mu$  having the correct behavior at  $\lambda = 0$  and  $\lambda = \infty$ . In the case where  $a$  has only simple roots we have only to prove that  $\mu$  is holomorphic on  $\Sigma - \{0, \infty\}$ , which is a weaker condition than  $\mu = f\nu + g$  with holomorphic function  $f, g$  defined on  $\mathbb{C}^*$  in the general case (when  $\nu$  has higher order roots,  $f$  could have poles). Keeping this in mind we define spectral data of a minimal cylinder by

**Definition 7.11.** *The spectral data of a minimal cylinder of finite type in  $S^2 \times \mathbb{R}$  consists of  $(a, b)$  where  $a$  is a complex polynomial of degree  $2g$ , and  $b$  is a complex polynomial of degree  $g + 1$  such that*

- (i)  $\lambda^{2g} \bar{a}(\bar{\lambda}^{-1}) = a(\lambda)$  and  $\lambda^{-g} a(\lambda) \leq 0$  for all  $\lambda \in S^1$
- (ii)  $\lambda^{g+1} \bar{b}(\bar{\lambda}^{-1}) = -b(\lambda)$
- (iii)  $b(0) = -\frac{\tau e^{i\Theta}}{32} \in e^{i\Theta/2} \mathbb{R}$

- (iv)  $\operatorname{Re} \left( \int_{\alpha_i}^{1/\bar{\alpha}_i} \frac{bd\lambda}{\nu\lambda^2} \right) = 0$  for all roots  $\alpha_i$  of  $a$  where the integral is computed on the straight segment  $[\alpha_i, 1/\bar{\alpha}_i]$ .
- (v) The unique function  $h : \tilde{\Sigma} \rightarrow \mathbb{C}$  where  $\tilde{\Sigma} = \Sigma - \cup \gamma_i$  and  $\gamma_i$  are closed cycles over the straight lines connecting  $\alpha_i$  and  $1/\bar{\alpha}_i$ , such that

$$\sigma^* h(\lambda) = -h(\lambda) \text{ and } dh = \frac{bd\lambda}{\nu\lambda^2}$$

takes values on  $i\pi\mathbb{Z}$  at all roots of  $(\lambda - 1)a$ .

- (vi) In the case when  $a$  has higher order roots there are holomorphic functions  $f, g$  defined on  $\mathbb{C}^*$  with  $e^h = f\nu + g$ .

**Corollary 7.12.** *If  $\xi_\lambda \in \mathcal{P}_g$  corresponds to a periodic minimal immersion  $X : \mathbb{C}/\tau\mathbb{Z} \rightarrow \mathbb{S}^2 \times \mathbb{R}$ , then there exists a polynomial  $b$ , which obeys (i)-(vi) with  $a(\lambda) = -\lambda \det(\xi_\lambda)$ .*

*If  $(a, b)$  obeys (i)-(vi), then all  $\xi_\lambda \in I(a)$  correspond to periodic minimal immersions.*

## 8. ISOSPECTRAL GROUP AND EMBEDDED ANNULI

We prove that embeddedness is an isospectral property. First recall that the maximum principle at infinity implies the following

**Proposition 8.1.** *Assume that there is an isospectral set  $I(a)$  with  $a$  having simple roots, and  $\tau \in \mathbb{C}$  such that the induced immersion  $X : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  of  $\xi_\lambda \in I(a)$  satisfies  $X(z + \tau) = X(z)$ . Then if  $X(\mathbb{C}/\tau\mathbb{Z})$  is embedded then any  $\hat{\xi}_\lambda \in I(a)$  induces an embedded annulus  $\hat{X}(\mathbb{C}/\tau\mathbb{Z})$ .*

*Proof.* By the maximum principle we know that embeddedness is a closed condition. If there is a sequence of annuli  $A_n$  converging on compact sets to a limit  $A_0$ , then if  $A_n$  is embedded, the limit is embedded. If not there is a point of  $A_0$  where we locally have at least two disks in  $A_0$  which are intersecting with transversal tangent plane. Then for  $n$  large enough we can observe two transversely intersecting disks converging to  $A_0$  and the property to be embedded is a closed property in  $I(a)$ .

We prove that embeddedness is an open property in  $I(a)$ . Consider a potential  $\xi_\lambda$  inducing an embedded annulus  $X(z)$ , and a potential  $\hat{\xi}_\lambda$  close to  $\xi_\lambda$  inducing an annulus  $\hat{X}(z)$  with the same period  $\tau$ . We prove that  $\hat{X}(z)$  is embedded.

Since the Iwasawa decomposition is a diffeomorphism, the map  $X : \mathbb{C}/\tau\mathbb{Z} \times I(a) \rightarrow \mathbb{S}^2 \times \mathbb{R}$  where  $X(z, \xi_\lambda)$  is the immersion induced by  $\xi_\lambda$  at a point  $z$ , is uniformly smooth on compact sets of  $\mathbb{C}/\tau\mathbb{Z}$ . Then for all  $\epsilon > 0$  there is  $R > 2\tau$  such that on  $\bar{B}(0, R) \times I(a)$  there exists  $\delta > 0$  with

$$\xi_\lambda, \hat{\xi}_\lambda \in I(a), \|\xi_\lambda - \hat{\xi}_\lambda\| < \delta \quad \text{implies} \quad \|X(z, \xi_\lambda) - X(z, \hat{\xi}_\lambda)\|_{\mathbb{C}^2} < \epsilon \quad \text{for } z \in \bar{B}(0, R).$$

Since  $I(a)$  has a structure of a  $g$ -dimensional manifold and the Iwasawa group action defines a chart around  $\xi_\lambda$ , there is a  $t \in \mathbb{C}^g$  such that  $\hat{\xi}_\lambda = \pi(t)\xi_\lambda$  and  $st \in \mathbb{C}^g$ , define a smooth deformation  $\pi(st)\xi_\lambda, s \in [0, 1]$  between  $\xi_\lambda$  and  $\pi(t)\xi_\lambda$ .

Locally in a neighborhood of some  $\xi_0 \in I(a)$  and some  $\delta > 0$ , there exists  $\delta_1 > 0$  and  $\delta_2 > 0$  such that  $\|\pi(t)\xi_\lambda - \xi_\lambda\| \leq \delta$  for  $\|\xi_\lambda - \xi_0\| < \delta_2$  and  $|t| < \delta_1$  and by compactness of  $I(a)$  there is a  $\delta' > 0$  such that for all  $\xi_\lambda \in I(a)$  and  $|t| < \delta'$ , we have

$$\|\pi(t)\xi_\lambda - \xi_\lambda\| \leq \delta.$$

Now consider annuli  $X(\mathbb{C}/\tau\mathbb{Z})$  and  $\hat{X}(\mathbb{C}/\tau\mathbb{Z})$  induced by potentials  $\xi_\lambda$  and  $\hat{\xi}_\lambda = \pi(t)\xi_\lambda$ , and with polynomial Killing fields  $\zeta(z) = \pi(z)\xi_\lambda$  and  $\hat{\zeta}(z) = \pi(z)\hat{\xi}_\lambda = \pi(z)\pi(t)\xi_\lambda$ . Since the action  $\pi(t) : I(a) \rightarrow I(a)$  is commuting, we have

$$\|\hat{\zeta}(z) - \zeta_\lambda(z)\| = \|\pi(z)\pi(t)\xi_\lambda - \pi(z)\xi_\lambda\| = \|\pi(t)[\pi(z)\xi_\lambda] - \pi(z)\xi_\lambda\| \leq \delta$$

This proves that the polynomial Killing fields  $\zeta_\lambda(z)$  and  $\hat{\zeta}_\lambda(z)$  are uniformly close in  $z \in \mathbb{C}/\tau\mathbb{Z}$ , and it remains to consider the corresponding immersions.

Let  $z_0 \in \mathbb{C}/\tau\mathbb{Z}$  and  $\exp(z\zeta_\lambda(z_0)) = F_\lambda^{z_0}(z)B_\lambda^{z_0}(z)$  and  $\exp(z\hat{\zeta}_\lambda(z_0)) = \hat{F}_\lambda^{z_0}(z)\hat{B}_\lambda^{z_0}(z)$  the Iwasawa decompositions. Then  $F_\lambda^{z_0}(z) = F_\lambda^{-1}(z_0)F_\lambda(z + z_0)$  and  $\hat{F}_\lambda^{z_0}(z) = \hat{F}_\lambda^{-1}(z_0)\hat{F}_\lambda(z + z_0)$  by the commuting formula of Lemma 7.5. Since  $\|\hat{\zeta}(z_0) - \zeta(z_0)\| \leq \delta$ , we have  $\|F_\lambda^{z_0}(z) - \hat{F}_\lambda^{z_0}(z)\| < \epsilon$  for  $z \in \bar{B}(0, R)$ . This implies that

$$\|I(z_0)\hat{F}_\lambda(z + z_0) - F_\lambda(z + z_0)\| < \epsilon \text{ for } z \in \bar{B}(0, R)$$

where  $I(z_0) = F_\lambda(z_0)\hat{F}_\lambda^{-1}(z_0)$  is an isometry of  $\mathbb{S}^2 \times \mathbb{R}$  depending on  $z_0$ . This proves that for any  $\epsilon > 0$  and any point  $z_0 \in \mathbb{C}/\tau\mathbb{Z}$ , there is a compact set  $K(z_0) \subset \mathbb{C}/\tau\mathbb{Z}$  such that there exists an isometry  $I(z_0)$  of  $\mathbb{S}^2 \times \mathbb{R}$ , with  $I(z_0)\hat{X}(z)$  uniformly  $\mathcal{C}^2$  close to  $X(z)$  on  $K(z_0)$

$$\sup_{z \in K(z_0)} \|I(z_0)\hat{X}(z) - X(z)\|_{\mathcal{C}^2} \leq \epsilon$$

By the maximum principle at infinity for minimal surfaces in manifolds with non-negative Ricci curvature, each minimal annulus has an embedded tubular neighborhood  $T_{\epsilon_0} = Y((\mathbb{C}/\tau\mathbb{Z}), ] - \epsilon_0, \epsilon_0[)$ . This estimate implies that  $I(z_0)\hat{X}(z)$  is a graph on  $X(z)$  for  $z \in K(z_0)$  and if  $\epsilon < \epsilon_0$ , this graph is embedded in  $\mathbb{S}^2 \times \mathbb{R}$ . This proves that  $\hat{X}(z)$  is embedded when  $\hat{\xi}_\lambda$  is close enough to  $\xi_\lambda$ , and thus embeddedness is an open property of  $I(a)$ .  $\square$

In the general case the action has several orbits. If an annulus is embedded in one orbit, all elements of the orbit induce embedded annuli. Higher orbits are bubbletons. The following proposition is used to reduce bubbletons.

**Proposition 8.2.** *Assume that there is an isospectral set  $I(a)$  with  $a(\lambda)$  having higher order roots  $\alpha_0, \dots, \alpha_n$  with  $|\alpha_i| \neq 1$  for  $i = 1, \dots, n$  and  $\tau \in \mathbb{C}$  such that the induced immersion  $X : \mathbb{C} \rightarrow \mathbb{S}^2 \times \mathbb{R}$  of  $\xi_\lambda \in I(a)$ , with  $\xi_{\alpha_i} \neq 0$  for  $i = 1, \dots, n$  satisfies  $X(z + \tau) = X(z)$ . Then if  $X(\mathbb{C}/\tau\mathbb{Z})$  is embedded then any  $\hat{\xi}_\lambda = \pi(t)\xi_\lambda \in I(a)$  induces an embedded annulus  $\hat{X}(\mathbb{C}/\tau\mathbb{Z})$ .*

*Proof.* In [8] we prove that any potential  $\xi_\lambda$  splits into an element of  $\mathbb{CP}^1 \times I(\tilde{a})$  where  $\tilde{a}$  is a polynomial with lower degree. There is a group action  $\pi : \mathbb{C}^g \times I(a) \rightarrow I(a)$ , but the stabilizer  $\Gamma_{\xi_\lambda}$  is no longer isomorphic to  $\mathbb{Z}^g$ . We prove in proposition B.2, that there is a subgroup action

$$\tilde{\pi} : \mathbb{C} \times I(a) \rightarrow I(a)$$

which acts transitively on the  $\mathbb{CP}^1$  factor, fixing the second factor of  $I(\tilde{a})$ . We have  $\tilde{\pi}(\beta)\xi_\lambda = (L(\beta), \tilde{\xi}_\lambda)$  where  $\beta \in \mathbb{CP}^1$ . This implies that the group action  $\pi$  gives different orbits in  $I(a)$ . Each orbit is characterized by the property that all its element have the same roots of the same orders at  $\alpha_1, \dots, \alpha_n$ , since the action  $\pi(t)\xi_\lambda$  preserves the order of roots of  $\xi_\lambda$ .

Suppose  $\xi_{0,\lambda}$  is a potential of an embedded annulus and assume it has no zeroes. Consider its orbit  $O = \{\xi_\lambda \in I(a); \xi_\lambda = \pi(t)\xi_{0,\lambda}\}$ . We prove that if  $\hat{\xi}_\lambda \in O$ , then it induces an embedded annulus. In fact  $\xi_\lambda$  is a higher order bubbleton. If the annulus of  $\xi_\lambda$  closes with period  $\tau$ , we prove in Proposition B.3 that all  $\hat{\xi}_\lambda \in O$  give  $\tau$ -periodic annuli.

Now since  $(t, \xi_\lambda) \rightarrow \pi(t)\xi_\lambda$  is continuous, we have for all  $\delta > 0$  and  $\xi_0 \in I(a)$ , that there exists  $\delta_1 > 0$  and  $\delta_2 > 0$  such that  $\|\pi(t)\xi_\lambda - \xi_\lambda\| \leq \delta$  for  $\|\xi_\lambda - \xi_0\| \leq \delta_2, |t| < \delta_1$ .

If  $\xi_\lambda$  has a root, then a subgroup of  $t \in \mathbb{R}^g$  acts trivially on  $\xi_\lambda$ . But the important point is that the action extends continuously to this subset. Since  $I(a)$  is compact, the covering by the set  $B(\xi_0, \delta_2)$  has a finite subcovering. Then there is  $\delta' > 0$  with

$$\|\pi(t)\xi_\lambda - \xi_\lambda\| \leq \delta \text{ for } \xi_\lambda \in I(a), |t| < \delta'.$$

Note that if  $\hat{\xi}_\lambda = \pi(t)\xi_{\lambda,0}$  induces a different annulus, then  $\|\pi(z)\pi(t)\xi_{\lambda,0} - \pi(z)\xi_{\lambda,0}\| \leq \delta$  for all  $|t| \leq \delta'$  and for all  $z \in \mathbb{C}$ . Translations in the annulus preserve the orbit because  $\pi(z)\xi_\lambda$  has a

root if and only if  $\xi_\lambda$  has a root. Therefore we see that  $\hat{\zeta}_\lambda(z)$  and  $\zeta_{0,\lambda}$  stay uniformly close and we conclude as in the preceding lemma that the orbit is a subset of  $I(a)$  inducing embedded annuli.  $\square$

**Remark 8.3.** *We remark that in the case where  $\xi_\lambda$  would have a root at  $\alpha_0$  then the group action  $\pi(t)\xi_\lambda$  would have a root too. Then at this particular point the group action does not generate a real  $g$ -dimensional manifold. But one can prove that  $\pi$  generates a  $g-2n$  dimensional subspace of the tangent space of  $\mathcal{P}_g$ . The subgroup action  $\tilde{\pi}(\beta)$  with  $\beta \in \mathbb{C}^n$  acts trivially on the set of potentials having  $2n$  roots counted with multiplicity. This is equivalent to the situation where one can remove the singularity without changing the immersion (see proposition 2.4).*

## 9. DEFORMATION OF FINITE TYPE ANNULI

In this section we describe how we deform a minimal cylinder of finite type by deforming the spectral curve  $\Sigma$  and preserving the closing condition. A minimal cylinder of  $\mathbb{S}^2 \times \mathbb{R}$  possesses a monodromy  $F_\lambda(\tau)$  whose eigenvalue  $\mu(\tau)$  is a holomorphic function on  $\Sigma$  with essential singularities at  $\lambda = 0$  and  $\lambda = \infty$ . In the following we deform  $\Sigma = \{\nu^2 = \lambda^{-1}a(\lambda)\}$  by moving the roots of the polynomial  $a$  without destroying the global properties of the holomorphic function  $\mu$ . The existence of  $\mu$  on  $\Sigma$  with

$$d \ln \mu = \frac{b d\lambda}{\nu \lambda^2}$$

which satisfy the closing condition of definition 7.11, assure that the spectral data  $(a, b)$  parameterize a closed annulus. Then we derive vector fields on open sets of spectral data

$$\mathcal{M} := \{(a, b)\}$$

and show that their integral curves are differentiable families of spectral data of periodic minimal cylinders.

We parameterize such deformations by one parameter  $t \in [0, \varepsilon)$ . Along the deformation we will increase the length of the period  $|\tau|$ . This will be a condition on the vector fields on the set  $(a(t), b(t))$ .

**The Whitham deformation.** We consider the function  $\ln \mu$  as a function depending on  $\lambda$  and  $t$ . On  $\Sigma$ , the multivalued holomorphic function  $\ln \mu$  has simple poles at  $\lambda = 0$  and  $\lambda = \infty$ . Let us define  $U_1, U_2, \dots, U_{2g}$  a covering by open subsets of  $\Sigma$ , such that each  $U_i$  contains at most one branch point  $\alpha_i$  and  $U_{2g+1}$  is an open neighborhood at  $(\infty, 0)$ ,  $U_{2g+2}$  an open neighborhood of  $(\infty, \infty)$ . We can write locally the meromorphic function on  $\Sigma$  by

$$\ln \mu = \begin{cases} f_i(\lambda)\sqrt{\lambda - \alpha_i} + \pi i n_i & \text{on } U_i, 1 \leq i \leq 2g \\ \lambda^{-1/2} f_{2g+1}(\lambda) + \pi i n_{2g+1} & \text{on } U_{2g+1} \\ \lambda^{1/2} f_{2g+2}(\lambda) + \pi i n_{2g+2} & \text{on } U_{2g+2}. \end{cases}$$

We can write locally on the open set  $U_i$ ,

$$\partial_t \ln \mu = \partial_t f_i(\lambda)\sqrt{\lambda - \alpha_i} - \frac{\dot{\alpha}_i f_i(\lambda)}{2\sqrt{\lambda - \alpha_i}}.$$

We remark that at each branch point  $\partial_t \ln \mu$  has a first order pole on  $\Sigma$ . Since the branches of  $\ln \mu$  differ from each other by an integer multiple of  $2\pi i$ , then  $\partial_t \ln \mu$  is single valued on  $\Sigma$  and can have poles only at the branch points of  $\Sigma$ , or equivalently at the zeroes of  $a$  and at  $\lambda = 0$  or  $\lambda = \infty$ . Collecting all these conditions we can write  $\partial_t \ln \mu$  globally on  $\Sigma$  by

$$(9.1) \quad \partial_t \ln \mu = \frac{c(\lambda)}{\nu \lambda}$$

with a real polynomial  $c$  of degree at most  $g + 1$ , which satisfies the reality condition

$$(9.2) \quad \lambda^{g+1} \overline{c(\bar{\lambda}^{-1})} = c(\lambda).$$

The abelian differential  $d \ln \mu$  of the second kind is of the form

$$(9.3) \quad d \ln \mu = \frac{b d\lambda}{\nu \lambda^2}$$

where  $b$  is a real polynomial of degree  $g + 1$  which satisfies the reality condition

$$\lambda^{g+1} \overline{b(\bar{\lambda}^{-1})} = -b(\lambda).$$

We differentiate (9.3) with respect to  $t$ , and (9.1) with respect to  $\lambda$  and derive

$$\begin{aligned} \partial_{t\lambda}^2 \ln \mu &= \partial_\lambda \frac{c}{\nu \lambda} = \frac{c'}{\nu \lambda} - \frac{c}{\nu \lambda^2} - \frac{c\nu'}{\nu^2 \lambda} = \frac{2\nu^2 \lambda^2 c' - 2\lambda \nu^2 c - ca'\lambda + ca}{2\nu^3 \lambda^3} \\ \partial_{\lambda t}^2 \ln \mu &= \partial_t \frac{b}{\nu \lambda^2} = \frac{\dot{b}}{\nu \lambda^2} - \frac{b\dot{\nu}}{\nu^2 \lambda^2} = \frac{2\nu^2 \lambda \dot{b} - b\dot{a}}{2\nu^3 \lambda^3} \end{aligned}$$

Hence second partial derivatives commute if and only if

$$(9.4) \quad -2\dot{b}a + b\dot{a} = -2\lambda ac' + ac + \lambda a'c.$$

Both sides in the last formula are polynomials of at most degree  $3g + 1$  which satisfy a reality condition. This corresponds to  $3g + 2$  real equations. Choosing a polynomial  $c$  which satisfies the reality condition (9.2) we thus obtain a vector field on polynomials  $\{(a, b)\}$ . When  $a$  and  $b$  have no common roots  $\alpha_i$  and  $\beta_i$  then equation (9.4) uniquely determines the values of  $\dot{a}$  at the roots of  $a$  and the values of  $\dot{b}$  at the roots of  $b$ . In the case, where  $a$  and  $b$  have only simple roots, we have the Whitham deformation system

$$(9.5) \quad \dot{a}(\alpha_i) = \frac{\alpha_i a'(\alpha_i) c(\alpha_i)}{b(\alpha_i)}; \quad \dot{b}(\beta_i) = \frac{2\beta_i a(\beta_i) c'(\beta_i) - a(\beta_i) c(\beta_i) - \beta_i a'(\beta_i) c(\beta_i)}{2a(\beta_i)}.$$

**Singularity of the Whitham deformation.** By defining such polynomials  $c$  we obtain vector fields on the space of polynomials  $a$  of degree  $2g$  and polynomials  $b$  of degree  $g + 1$  satisfying the reality condition. The value of  $c$  at each  $\alpha_i$  and  $\beta_i$  determine uniquely a tangent vector  $(\dot{a}, \dot{b})$  on the space moduli of minimal cylinder. Equations (9.5) define rational vector fields on the space of spectral data  $(a, b)$  of minimal annuli. We will study in the next subsection the condition on  $c$  to preserve the period along the deformation when  $a$  and  $b$  have no common roots.

The integral curves have possible singularities when  $a$  and  $b$  have common roots but we will see below that we can pass through such singularities. The closing conditions on  $c$  are the same, but the difference now is that the vector field has a meromorphic singularity. In this setting we reparameterize the integral curves  $(a(t), b(t))$  in the neighborhood of the singularity by using a smooth reparametrization of the flow using the value at the branch point of the function  $\ln \mu$ . We construct a smooth model for the space of spectral curves preserving  $\mu$  in the spirit of deformations of branched coverings of Hurwitz in section 9.2. In this model we reparameterize the equation (9.5), and obtain instead of a pole a zero in the induced vector field. At the roots of the vector field in the equation (9.5) we show that the linearization has non-zero positive and negative eigenvalues and thus integral curve moving in and out of the singularity. This give us a way to pass through common roots of  $a$  and  $b$  in the meromorphic equation (9.5) (see theorem 9.5).

**Opening double points.** Another application of this smooth model is a construction of integral curves which open double points on  $\Sigma$  and increase the genus of the spectral curve. To open a double point we add a double zero to the polynomial  $a(\lambda)$  and a simple zero to the polynomial  $b(\lambda)$ , and note that this does not change  $\mu$ . Thus we create a common root of  $a$  and

*b.* Passing through the singularity means that we open this double point. We can open double points only at points where  $\mu = \pm 1$ . When two roots coalesce, this is a double point of  $\mu$ .

There is a difference between opening double points on  $\mathbb{S}^1$  and opening pairs of double points away from  $\mathbb{S}^1$ . In the first case, there exists only one direction which preserves the condition  $\lambda^{-g}a(\lambda) \leq 0$  for  $\lambda \in \mathbb{S}^1$ . The other direction should move the points on the unit circle, a deformation which is not allowed. Pairs of double points can be opened in both directions. We can pass through such singularities and this defines a deformation which increases the genus of the spectral curve.

To increase the flux simultaneously while opening a non-real double point we choose a  $c$  which changes the flux. At the double point, we find an integral curve of the equation and we pass through the singularity. Since for non-real double points we can open double points in both direction of the integral curve, there exists a direction which increases the flux and opens the double point.

In the case where we open a real double point, there is only one direction which preserves the condition  $\lambda^{-g}a(\lambda) \leq 0$  and leads away from the unit circle, preserving the reality condition. In general we cannot increase the flux by opening real double points. This is the case where we open double points on the genus zero curve with the period closing the flat embedded cylinder (see section C.2).

**Increasing the flux along the deformation.** At  $\lambda = 0$ , we have  $-2\dot{b}(0)a(0) + b(0)\dot{a}(0) = a(0)c(0)$ . This equation can be expressed as

$$\partial_t(b\sqrt{-a}^{-1})(0) = \frac{-2\dot{b}(0)a(0) + b(0)\dot{a}(0)}{2(-a(0))^{3/2}} = \frac{-c(0)}{2\sqrt{-a(0)}}$$

Then

$$-2\operatorname{Re} \partial_t(\ln b\sqrt{-a}^{-1})(0) = -2\partial_t \ln |\tau| = \operatorname{Re} \frac{c(0)}{b(0)}.$$

**Lemma 9.1.** *If  $g > 0$ , then there exists a  $c(\lambda)$  preserving the closing condition and reality condition which integrates  $|\tau|$  along its integral curves.*

*Proof.* We choose independently of the fact that we have a singularity in the flow a vector field with  $\operatorname{Re} \frac{c(0)}{b(0)} < 0$ . This condition assures that we increase the third coordinate of the flux along the deformation. Choose  $c(\lambda) = (\lambda - 1)(b(0) - \bar{b}(0)\lambda^g)$  and check that  $c(0) = -b(0)$ . By appendix C when  $g = 0$  then  $b(0) = \frac{-\pi}{16} \in \mathbb{R}$  and  $c = 0$ .  $\square$

**9.1. Flow without common roots of  $a$  and  $b$ .** In the case where  $b$  has higher order roots  $\beta_j$  of order  $k$ , we need to consider the system (9.5), with additional equations  $\dot{b}'(\beta_j), \dots, \dot{b}^{(k-1)}(\beta_j)$ . The equations can be easily derived from the derivatives of (9.4), taking into account that  $b(\beta_j) = b'(\beta_j) = \dots = b^{(k-1)}(\beta_j) = 0$ .

**Theorem 9.2.** *Let  $U$  be an open subset of spectral data  $(a, b)$  in  $\mathcal{M}_g$ , with  $a, b$  having no common roots. Then a polynomial  $c$  of degree  $g + 1$  which satisfies the reality condition (9.2) exists, and the equations (9.4) define a smooth vector field on  $U$ .*

*If (i)  $c(1) = 0$ , and (ii)  $\operatorname{Im}(c(0)/b(0)) = 0$ , then integral curves  $(a(t), b(t)) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$  are spectral data of a continuous family of minimal annuli in  $\mathbb{S}^2 \times \mathbb{R}$ . If  $\operatorname{Re}(c(0)/b(0)) < 0$ , the curve  $(a(t), b(t))$  is the spectral data of minimal annuli with flux bounded away from 0.*

*Proof.* The spectral data  $(a, b)$  satisfy hypothesis of Definition 7.11. The solutions  $\dot{a}$  and  $\dot{b}$  of equation (9.4) are rational expressions of the coefficients of  $a, b$  and  $c$ . If  $a$  and  $b$  have no common roots, then the Taylor coefficients of  $\dot{a}$  and  $\dot{b}$  at the roots of  $a$  and  $b$  up to the order

of the roots minus one, respectively, and the highest coefficient of  $\dot{b}$  are uniquely determined by equation (9.5). Hence in this case the denominators of the rational expressions for  $\dot{a}$  and  $\dot{b}$  do not vanish. Hence for smooth  $c(\lambda)$  the corresponding  $\dot{a}$  and  $\dot{b}$  are smooth too. We can thus find an integral curve in the set of complex polynomial.

The transformation rules of  $\mu$  under  $\sigma$ ,  $\rho$  and  $\eta$  are preserved under the flows of the vector field corresponding to  $c$ . Hence the integrals of  $d \ln \mu$  along any smooth path from one root of  $a$  to another root of  $a$  is preserved too. This implies that the subset of spectral data  $(a, b)$ , which determine a single valued function  $\mu$  with  $\sigma^* \mu = \mu^{-1}$ ,  $\rho^* \mu = \bar{\mu}^{-1}$  and  $\eta^* \mu = \bar{\mu}$  is preserved under this flow.

Due to (9.1)  $\partial_t \ln \mu$  is a meromorphic function on the hyperelliptic curve  $\Sigma$  and the periods of the meromorphic differential  $d \ln \mu$  do not depend on  $t$ . The closing condition of definition 7.11 is preserved if  $\frac{d}{dt} \ln \mu(\alpha_i(t), t) = 0$  for all roots  $\alpha_i(t)$  of  $a(t)$ , which holds precisely when

$$d \ln \mu(\alpha_i(t), t) \partial_t \alpha_i + \partial_t \ln \mu(\alpha_i(t), t) = 0.$$

Using equations (9.1) and (9.3), the closing conditions are therefore preserved if and only if

$$(9.6) \quad \dot{\alpha}_i = \frac{-\alpha_i c(\alpha_i)}{b(\alpha_i)}$$

This equation describes the deformation of the spectral curve  $\Sigma$ . The polynomial  $b$  exists and integrals along cycle are multiple of  $i\pi\mathbb{Z}$ . The condition  $c(1) = 0$  ensures that the value of  $\mu$  remains constant at  $\lambda = 1$ , that is  $\mu(1) = \pm 1$ .

To close the annulus, we have to keep coefficient  $b(0) \in e^{i\Theta/2}\mathbb{R}$  where  $a(0) = \frac{-e^{i\Theta}}{16}$  for each  $t \in \mathbb{R}$ . At  $\lambda = 0$ , we observe that  $-2\dot{b}(0)a(0) + b(0)\dot{a}(0) = a(0)c(0)$  and as in the computation of increasing flux  $\partial_t(\ln b\sqrt{-a}^{-1})(0) = \frac{-c(0)}{2b(0)}$ . To close the period we need that  $b(0)\sqrt{-a(0)}^{-1} = \frac{\tau e^{i\Theta/2}}{8} \in \mathbb{R}$  along the deformation. This gives condition (ii).

In order to control the flux, we need to control the period  $\tau$  away from zero. This explains the condition on  $\operatorname{Re} \frac{c(0)}{b(0)} < 0$  along the deformation (see lemma 9.1).  $\square$

**9.2. Smooth parametrization of spectral data and opening double point.** We consider a non-singular spectral curve  $\Sigma$  of genus  $g$  ( $a$  has only simple roots) and spectral data of a minimal annulus given by  $(a, b) \in \mathcal{M}_g$ . We consider  $\mu = f\nu + g$  (see definition 7.11) with  $g^2 - f^2\nu^2 = 1$  (using the hyperelliptic involution). Roots of  $f$  are points where  $\mu^2 = 1$ . These points are called double points and possibly a higher genus spectral curve can be desingularized at this point. In this section we consider the reverse possibility, that is we are interested in deformations of  $(a, b)$  in a moduli space  $\mathcal{M}_{g+n}$  for  $n$  large enough. This possible deformation occurs when we want to pass through a singularity of the vector field  $c$ , that is when  $a$  and  $b$  have common roots or when we want to increase the genus of a spectral curve  $\Sigma$  by opening a double point.

We remark that when  $a$  and  $b$  have common roots at  $\alpha_0$ , by integrating  $d \ln \mu = \frac{bd\lambda}{\nu\lambda}$  at  $\alpha_0$ , we have  $\ln \mu = f_0(\lambda)(\lambda - \alpha_0)^d + g_0$  with  $d \geq 3/2$ ,  $f_0(\alpha_0) \neq 0$  and  $f(\lambda) = f_0(\lambda)(\lambda - \alpha_0)^{d-1/2}$ . We are in the situation where  $f(\alpha_0) = 0$ .

In the set of points where  $\mu(\alpha_0) = \pm 1$ , we have  $a(\alpha_0) = 0$  or  $f(\alpha_0) = 0$ . In the case where both are zero, this is the case of common roots of  $a$  and  $b$ . When only  $f(\alpha_0) = 0$  and  $a(\alpha_0) \neq 0$ , this is a simple double point (which is not a branch point). There exist infinitely many such double points. In the last case, when  $a(\alpha_0) = 0$  and  $f(\alpha_0) \neq 0$ , we have a simple branch point of  $\Sigma$ .

We consider spectral data  $(\tilde{a}, \tilde{b}) = (ap^2, bp)$ , such that  $\mu$  can be written as  $\mu = \tilde{f}\tilde{\nu} + \tilde{g}$  with holomorphic functions  $\tilde{f}$  and  $\tilde{g}$  on  $\mathbb{C}^*$ , with  $\tilde{\nu}^2 = \frac{\tilde{a}(\lambda)}{\lambda}$  and  $\tilde{f}$  does not vanish at the roots of  $\tilde{a}$ . This means that we add to  $a$  an appropriate number of double roots at all common roots of  $a$

and  $b$ . Moreover we have the choice to do the same at finitely many double points. Now we construct a smooth parametrization of  $(\tilde{a}, \tilde{b})$  in  $\mathcal{M}_{g+n}$ .

We choose simply connected neighbourhoods  $V_1, \dots, V_M$  in  $\mathbb{C}^*$  at all roots of  $\tilde{b}$  including the common roots with  $\tilde{a}$ . Let  $U_1, \dots, U_M$  denote the pre-images in  $\Sigma \setminus \{0, \infty\}$  of  $V_1, \dots, V_M$  under the map  $\lambda$ . For  $m = 1, \dots, M$  we choose on  $U_m$  a branch of the function  $\ln \mu$ . On  $U_m$  the function  $\frac{1}{2\pi i}(\ln \mu + \sigma^* \ln \mu)$  is equal to a constant integer  $n_m$  at the branch point. These branches obey

$$(9.7) \quad (\ln \mu - n_m i\pi)^2 = A_m \quad \text{for } m = 1, \dots, M,$$

with holomorphic function  $A_m$  on  $V_m$  which vanish at the roots of  $\tilde{a}$ . Since  $\sigma^*(\ln \mu - n_m i\pi) = -(\ln \mu - n_m i\pi)$ , the function  $A_m$  depends only on  $\lambda$  (see Forster [6], theorem 8.2). If we choose  $U_m$  and  $V_m$  small enough, then the derivative of  $A_m$  has no roots besides the corresponding root of  $\tilde{b}$  ( $d \ln \mu$  vanishes at roots of  $b$ ). The roots of  $\tilde{b}$  are exactly the roots of the derivative of  $A_m$ . For small enough  $U_m$  and  $V_m$  there exists a biholomorphic map  $\lambda \mapsto z_m(\lambda)$  from  $V_m$  to a simply connected open neighbourhood  $W_m$  of  $0 \in \mathbb{C}$ , such that  $A_m$  coincides with

$$(9.8) \quad A_m(\lambda) = z_m^{d_m}(\lambda) + a_m$$

At a root of  $\tilde{b}$ , which is not a root of  $\tilde{a}$  (i.e. this is a root of  $b$  which is not a root of  $a$ , and so a root of  $d \ln \mu$ ), the constant  $a_m \neq 0$ , and  $d_m - 1$  is the order of the roots of  $b$ . At a common root of  $\tilde{a}$  and  $\tilde{b}$ , the constant  $a_m = 0$  and  $d_m$  is an odd integer in the case of common roots of  $a$  and  $b$  and even in the case of double points.

We describe spectral curves in a neighbourhood of the given spectral curve by small perturbations  $\tilde{A}_1, \dots, \tilde{A}_M$  of the polynomials  $A_1, \dots, A_M$ . More precisely, we consider polynomials  $\tilde{A}_1, \dots, \tilde{A}_M$  of the form

$$(9.9) \quad \tilde{A}_m(z_m) = z_m^{d_m} + \tilde{a}_{m,1} z_m^{d_m-1} + \tilde{a}_{m,2} z_m^{d_m-2} + \dots + \tilde{a}_{m,m}$$

with coefficients  $\tilde{a}_{m,2}, \dots, (\tilde{a}_{m,m} - a_m)$  nearby zero. By a shift  $z \rightarrow z + z_0$ , we can always assume that the sum of the roots is zero and then  $\tilde{a}_{m,1} = 0$ . We glue each  $W_m$  of the sets  $W_1, \dots, W_M$  to  $\mathbb{P}^1 \setminus (V_1 \cup \dots \cup V_M)$  along the boundary of  $V_m$  in such a way that for all  $m = 1, \dots, M$  the polynomial  $\tilde{A}_m$  coincides with the unperturbed function  $A_m$  in a tubular neighborhood of the boundary  $\partial W_m$ . We obtain a new copy of  $\mathbb{CP}^1$ . By uniformization, there exists a new global parameter  $\tilde{\lambda}$ , which is equal to 0 and  $\infty$  at the two points corresponding to  $\lambda = 0$  and  $\lambda = \infty$ , respectively. This new parameter is unique up to multiplication with elements of  $\mathbb{C}^*$ . There exists a biholomorphic map  $\tilde{\lambda} = \phi(\lambda)$  which changes the parameter  $\lambda \in \mathbb{CP}^1 \setminus (V_1 \cup \dots \cup V_M)$  in the global parameter  $\tilde{\lambda}$ . There is a biholomorphic map  $\tilde{\lambda} = \phi_m(z_m)$  which changes the local parameter  $z_m \in W_m$  into  $\tilde{\lambda}$ . Let  $\tilde{\lambda} \rightarrow \tilde{a}(\tilde{\lambda})$  be the polynomial whose roots (counted with multiplicities) coincide with the zero set of  $\tilde{A}_1(\tilde{\lambda}), \dots, \tilde{A}_M(\tilde{\lambda})$  and the roots of  $\tilde{\lambda} \rightarrow a \circ \phi^{-1}(\tilde{\lambda})$  on  $\mathbb{P}^1 \setminus (V_1 \cup \dots \cup V_M)$ .

Now  $\tilde{\Sigma} = \{(\tilde{\nu}, \tilde{\lambda}) \in \mathbb{C}^2 \mid \nu^2 = \frac{\tilde{a}(\tilde{\lambda})}{\tilde{\lambda}}\}$  is a new hyperelliptic Riemann surface associated to the set of polynomials  $\tilde{A}_1, \dots, \tilde{A}_M$ . We say that polynomials  $\tilde{A}_1, \dots, \tilde{A}_M$  respect the reality condition if the involutions  $\sigma, \rho$  and  $\eta$  lift to involutions of  $\tilde{\Sigma}$  and then define a spectral curve. In this case, the parameter  $\lambda$  is determined up to a multiplication of unimodular numbers. We call this transformation a Möbius transformation. The equations

$$(9.10) \quad (\ln \mu - n_m i\pi)^2 = \tilde{A}_m(\tilde{\lambda}) = \tilde{A}_m \circ \phi_m^{-1}(\tilde{\lambda}) = \tilde{A}_m(z_m) \quad \text{for } m = 1, \dots, M$$

define a function  $\mu$  on the pre-image of  $\phi_m(W_m) \cap \mathbb{P}^1$  by the map  $\tilde{\lambda}$  into  $\tilde{\Sigma}$ . The function  $\mu$  extends to the pre-image of  $\mathbb{C}^* \setminus (V_1 \cup \dots \cup V_M)$  by  $\tilde{\lambda} = \phi(\lambda)$  and coincides with the unperturbed  $\mu$  on this set.



On  $\tilde{\Sigma}$  the differential  $d \ln \mu$  is again meromorphic and takes the form  $d \ln \mu = \frac{\tilde{b}(\tilde{\lambda}) d\tilde{\lambda}}{\tilde{\nu} \tilde{\lambda}^2}$  with a unique polynomial  $\tilde{b}$ . By taking the derivative of (9.10) we have

$$2(\ln \mu - n_m \pi i) \frac{d}{d\lambda} \ln \mu = \tilde{A}'_m(z_m(\tilde{\lambda})) z'_m(\tilde{\lambda}).$$

Then roots of  $\tilde{b}$  are the roots of the derivatives of  $\tilde{A}_1, \dots, \tilde{A}_M$ .

**Proposition 9.3.** *The set of polynomials  $\tilde{A}_1, \dots, \tilde{A}_M$  which respect the reality condition and coefficients  $\tilde{a}_{m,2}, \dots, (\tilde{a}_{m,m} - a_m)$  nearby zero define spectral data of periodic solutions of the sinh-Gordon equation in a neighbourhood of  $\Sigma$ . The spectral genus is generically  $g + n$ . A  $\tilde{c}$  depending smoothly on  $(\tilde{a}, \tilde{b})$  defines a smooth vector field on these parameters. Integral curves of  $\tilde{c}$  in the set of spectral data deform  $\tilde{A}_1, \dots, \tilde{A}_M$ .*

**Remark 9.4.** *By definition 7.11, the existence of the function  $\mu$  assures the periodicity of the polynomial Killing field with initial value  $\tilde{\xi}_\lambda \in I(\tilde{a})$ . If  $\tilde{c}(1) = 0$  and  $\text{Im}(\tilde{c}(0)/\tilde{b}(0)) = 0$ , the deformation preserves the closing condition of the surface along the integral curves defined by  $\tilde{c}(\tilde{\lambda})$ . The condition  $\text{Re}(\tilde{c}(0)/\tilde{b}(0)) < 0$  increase the length of the flux  $|\tau|$ .*

*Proof.* First given a smooth family of  $\tilde{A}_1(t), \dots, \tilde{A}_M(t)$  which respect the reality condition, we calculate the change of  $\ln \mu$  and the uniquely defined corresponding polynomial  $\tilde{c}(\tilde{\lambda})$ . In a second step, we will prove that a given polynomial  $\tilde{c}(\tilde{\lambda})$  will smoothly determine a vector field  $\dot{\tilde{A}}_1, \dots, \dot{\tilde{A}}_M$ . If the polynomial  $A_m$  changes, then also the biholomorphic map  $\lambda \mapsto z_m(\lambda)$  changes. If we differentiate (9.10) with  $z_m(\tilde{\lambda}) = \phi_m^{-1}(\tilde{\lambda})$  we obtain

$$2(\ln \mu - n_m \pi i) \frac{d}{dt} \ln \mu = \dot{\tilde{A}}_m(z_m(\tilde{\lambda})) + \tilde{A}'_m(z_m(\tilde{\lambda})) \dot{z}_m(\tilde{\lambda})$$

The equations (9.1) and (9.3) imply

$$(9.11) \quad \frac{\tilde{\lambda} \tilde{c}(\tilde{\lambda})}{\tilde{b}(\tilde{\lambda})} = \frac{\dot{\tilde{A}}_m(z_m(\tilde{\lambda}))}{\tilde{A}'_m(z_m(\tilde{\lambda})) \dot{z}_m(\tilde{\lambda})} + \frac{\dot{z}_m(\tilde{\lambda})}{z'_m(\tilde{\lambda})}.$$

On the right hand side the second term has no poles at the roots of  $b$  since  $z_m(\tilde{\lambda})$  is biholomorphic on  $V_m$  (a derivative of a biholomorphic map doesn't vanish). The singular parts of the meromorphic function on the left hand side at the roots of  $\tilde{b}$  are determined by the first term on the right hand side. The family  $\tilde{A}_1(t), \dots, \tilde{A}_M(t)$  and the finitely many values of the derivatives of  $z_m(\tilde{\lambda})$  at the roots of  $\tilde{b}(\tilde{\lambda})$  determine the values of  $\tilde{\lambda} \rightarrow \tilde{c}(\tilde{\lambda})$  at all roots of  $\tilde{b}$ . The polynomial  $\tilde{c}$  is determined up to a multiple constant of  $\tilde{b}$  by a purely imaginary complex number ( $\tilde{c} \rightarrow \tilde{c} + i\gamma b$ ). This degree of freedom comes from the Möbius transformation on the parameter  $\tilde{\lambda}$ . If  $\tilde{c} = i\gamma b$ , then  $\tilde{\lambda} \rightarrow e^{i\gamma t} \tilde{\lambda}$ . The choice of the parameter  $\tilde{\lambda}$  fixes the degree of freedom in the polynomial  $\tilde{c}$ .

Conversely, the choice of  $\tilde{c}$  determines the value of  $\dot{\tilde{A}}_m$  at all the zeroes of  $\tilde{b}$  which coincide with the zeroes of  $\tilde{A}'_m$ . By definition of  $\tilde{A}_m$ , the degree of  $\dot{\tilde{A}}_m$  is less than the degree of  $\tilde{A}'_m$  (The two highest coefficients are independent of  $t$ ). The family  $\dot{\tilde{A}}_m$  depends linearly on  $\tilde{c}$ .  $\square$

**9.3. Flow with common roots of  $a$  and  $b$ .** In this case the vector field defined by polynomials  $c$  and equation (9.5) has singularities. We study this situation and describe how to continuously extend through such singularities. To do that we consider the embedding of  $\mathcal{M}_g$  into  $\mathcal{M}_{g+n}$  with large enough  $n$ . We assume in this section that  $a$  has only simple roots. We use in this situation the parametrization of spectral data of  $\mathcal{M}_{g+n}$  described in section 9.2. Common roots of  $a$  and  $b$  should be considered as higher order roots of  $\tilde{a}$ . (We define  $(\tilde{a}, \tilde{b}) = (ap^2, bp)$  where  $p$  is the polynomial whose roots coincide with roots of  $f$  ( $f$  is define in definition 7.11 (vi))

counted with multiplicity at common roots of  $a$  and  $b$ . Then  $f = p\tilde{f}$ ,  $\tilde{f}$  has no roots at common roots of  $a$  and  $b$ .

The order  $d_m = 2\ell_m + 1$  is odd and at least three. The genus of the spectral curve is preserved, if the number of odd order roots of  $\tilde{a}$  is preserved (see below). This is equivalent to the condition that all  $\tilde{A}_m$  of odd degree have only one odd order roots. Hence we consider the polynomials  $\tilde{A}_m$  of the form

$$\begin{aligned}\tilde{A}_m(z_m) &= (z_m - 2\alpha_m)p_m^2(z_m) \quad \text{with} \\ p_m(z_m) &= z_m^{\ell_m} + \beta_{m,1}z_m^{\ell_m-1} + \dots + \beta_{m,\ell_m} \quad \text{with } \beta_{m,1} = \alpha_m\end{aligned}$$

Since the sum of the zeroes of  $\tilde{A}_m$  equals zero, the odd order root is  $\beta_{m,1} = \alpha_m$  in the formula. In this case, all double roots of  $\tilde{a}$  will produce double roots of the spectral curves and will not contribute to the geometric genus of the spectral curve, so we remain in the space  $\mathcal{M}_g$  of spectral curves of genus  $g$ , removing  $n$  singularities in spectral curve of  $\mathcal{M}_{g+n}$ . In this parametrization double roots of  $A_m$  are simple roots of  $b$ . Then we can reduce the genus of the spectral curve without changing the function  $\mu$ . Hence we parameterize the coefficients of  $\tilde{A}_m$  by the coefficients  $\alpha_m = \beta_{m,1}$  and  $\beta_{m,2}, \dots, \beta_{m,\ell_m}$ .

**Proposition 9.5.** *Let  $(a, b)$  be the spectral data of a periodic solution of the sinh-Gordon equation. Suppose  $a$  and  $b$  have common roots. Assume  $c$  is a polynomial of degree  $g + 1$  such that  $\lambda^{g+1}\overline{c(1/\bar{\lambda})} = c(\lambda)$  ( $c$  can depend on  $a$  and  $b$  smoothly and does not vanish at  $(a_0, b_0)$ ) that does not vanish at the common roots of  $a$  and  $b$ . Then there exists for some  $\epsilon > 0$  a continuous integral curve  $(a_t, b_t)_{t \in (-\epsilon, \epsilon)}$  with  $(a_0, b_0) = (a, b)$  along the vector field induced by  $c$ , which is smooth on  $(-\epsilon, \epsilon) \setminus \{0\}$ . Moreover for  $t \in (-\epsilon, \epsilon) \setminus \{0\}$ ,  $a_t$  and  $b_t$  have no common roots.*

*Proof.* For a given  $c$ , we consider the smooth vector field  $\tilde{c}$  with  $c/b = \tilde{c}/\tilde{b}$  in the set of spectral data  $(\tilde{a}, \tilde{b})$  in  $\mathcal{M}_{g+n}$ . This definition assures that along any integral curve of  $\tilde{c}$  in  $\mathcal{M}_{g+n}$ , all polynomials  $\tilde{A}_m$  keep the form  $\tilde{A}_m = (z - 2\alpha_m)p_m^2(z)$  where  $\alpha_m$  is a moving branch point along an integral curve of the vector field  $\tilde{c}$  nearby a common root of  $a$  and  $b$  located at zero in this coordinate. In the following we will parameterize integral curves by moving the roots  $\alpha_1, \dots, \alpha_m$  and coefficients of  $p_1, \dots, p_m$  polynomials of degree  $\ell_1, \dots, \ell_m$ .

At  $t = 0$ , the vector field  $\tilde{c}/\tilde{b}$  has a singularity. We multiply the vector field with a real function depending on the coefficient of  $a$  and  $b$  which vanishes at common roots of  $a$  and  $b$  in such a way that the new vector field becomes smooth and without poles at  $t = 0$ . The new vector field will have a root at  $t = 0$ . We want to find a trajectory moving into the root of the vector field and a trajectory moving out of the root. For this purpose, we calculate the first derivative of the vector field and find non trivial stable and unstable eigenspaces. By the Stable Manifold Theorem (see e.g Teschl [25]) there then exist integral curves moving in and out the zero of the vector field.

We have to collect different common roots of  $a$  and  $b$  and prove that the linearized vector field has non-empty stable and unstable eigenspaces. This means that the first derivative of the vector field at a common root of  $a$  and  $b$  has non-zero eigenvalues with positive and negative real parts. We compute the first derivative at common roots  $\alpha_1, \dots, \alpha_m$  separately and coefficients of  $p_1, \dots, p_m$  at  $t = 0$ .

In  $V_m$  we consider the polynomial  $A_m(z_m)$ . We drop the subscript  $\bullet_m$  in the following and we consider  $A(z) = (z - 2\alpha)p^2(z) := \alpha^{2\ell+1} \left(\frac{z}{\alpha} - 2\right) q^2\left(\frac{z}{\alpha}\right)$  where  $\alpha \in \mathbb{C}$  is a complex value close to zero, and  $\ell := \ell_m$  is the degree of the polynomial  $q$ . The polynomial  $q$  depends on  $t$  and  $\alpha(t)$  with  $\alpha(0) = 0$  and  $\alpha^\ell(t)q\left(\frac{z}{\alpha(t)}\right) \rightarrow z^\ell$  when  $t \rightarrow 0$ .

We denote  $w = \frac{z}{\alpha}$  and we set  $q(w) = w^\ell + w^{\ell-1} + \gamma_{m,2}w^{\ell-2} + \gamma_{m,3}w^{\ell-3} + \dots + \gamma_{m,\ell}$ . If  $\alpha = 0$ , then  $A(z) = z^{2\ell+1}$  independently of the choice of the polynomial  $q$ . Hence we can choose freely

an appropriate  $q$  at  $t = 0$ . We should choose  $q$  at  $t = 0$  in such a way that the first derivative  $\frac{\dot{q}}{\alpha}$  is bounded at  $t = 0$ . The choice of  $q$  at  $t = 0$  is specified in Lemma 9.6.

$$\begin{aligned} \dot{A}(z) &= \dot{\alpha} \alpha^{2\ell} q\left(\frac{z}{\alpha}\right) \left( (2\ell+1) \left(\frac{z}{\alpha} - 2\right) q\left(\frac{z}{\alpha}\right) - \left(\frac{z}{\alpha}\right) q\left(\frac{z}{\alpha}\right) - 2 \left(\frac{z}{\alpha}\right) \left(\frac{z}{\alpha} - 2\right) q'\left(\frac{z}{\alpha}\right) \right) \\ &\quad + 2\alpha^{2\ell+1} q\left(\frac{z}{\alpha}\right) \left(\frac{z}{\alpha} - 2\right) \dot{q}\left(\frac{z}{\alpha}\right) \\ A' &= \alpha^{2\ell} q\left(\frac{z}{\alpha}\right) \left( q\left(\frac{z}{\alpha}\right) + 2 \left(\frac{z}{\alpha} - 2\right) q'\left(\frac{z}{\alpha}\right) \right) \\ \frac{\dot{A}}{A'} &= \frac{\dot{\alpha} \left( (2\ell+1)(w-2)q(w) - wq(w) - 2w(w-2)q'(w) \right) + 2\alpha(w-2)\dot{q}(w)}{q(w) + 2(w-2)q'(w)} \end{aligned}$$

Now  $A'$  has  $2\ell$  roots. Besides the  $\ell$  double roots of  $A$  it has  $\ell$  additional roots, which are equal to the roots of the polynomial

$$\frac{\alpha^\ell}{2\ell+1} \left( q\left(\frac{z}{\alpha}\right) + 2 \left(\frac{z}{\alpha} - 2\right) q'\left(\frac{z}{\alpha}\right) \right).$$

This polynomial has highest coefficient 1. Locally in  $V_m$ , the function  $\frac{\tilde{\lambda}\tilde{c}(\tilde{\lambda})}{b(\tilde{\lambda})}$  may be uniquely decomposed into a rational function depending on  $z \in \mathbb{C}$ , which vanishes at  $z \rightarrow \infty$  and a holomorphic function  $h(z)$  nearby the roots of  $b$ . Hence there exists a unique polynomial  $C(z)$  of degree  $\ell-1$ , such that in  $V_m$ , the Laurent decomposition gives

$$\frac{\tilde{\lambda}\tilde{c}}{b} = \frac{C(z)}{\frac{\alpha^\ell}{2\ell+1} \left( q\left(\frac{z}{\alpha}\right) + 2 \left(\frac{z}{\alpha} - 2\right) q'\left(\frac{z}{\alpha}\right) \right)} + h(z).$$

Therefore we obtain the differential equation in  $V_m$

$$\frac{\dot{A}(z)}{A'(z)} = \frac{C(z)}{\frac{\alpha^\ell}{2\ell+1} \left( q\left(\frac{z}{\alpha}\right) + 2 \left(\frac{z}{\alpha} - 2\right) q'\left(\frac{z}{\alpha}\right) \right)}.$$

This equation is equivalent to

$$\dot{\alpha} \left( (2\ell+1)(w-2)q(w) - wq(w) - 2w(w-2)q'(w) \right) + 2\alpha(w-2)\dot{q}(w) = \frac{(2\ell+1)C(\alpha w)}{\alpha^\ell}.$$

We have  $\alpha^\ell q\left(\frac{z}{\alpha}\right) \rightarrow z^\ell$  as  $t \rightarrow 0$  and at the starting point, we should choose  $q$  in such a way, that  $\frac{\dot{q}}{\alpha}$  stays bounded. If we do that then we can neglect the second term on the left hand side of the equation which is equivalent to  $\alpha$ . On the right hand side we have  $C(0) \neq 0$  and at the starting point, we choose a polynomial  $q$  with  $(2\ell+1)(w-2)q(w) - wq(w) - 2w(w-2)q'(w)$  constant. Therefore  $q$  is given by the polynomial in the lemma 9.6.

**Lemma 9.6.** *For each  $\ell \in \mathbb{N}$  there exists a unique polynomial  $\hat{q}$  of degree  $\ell$  with highest coefficients 1, such that*

$$(2\ell+1)(w-2)\hat{q}(w) - w\hat{q}(w) - 2w(w-2)\hat{q}'(w) = K$$

where  $K$  is a constant. This polynomial is the polynomial part of  $w^\ell \left(1 - \frac{2}{w}\right)^{-1/2}$ .

*Proof.* The equation  $(2\ell+1)(w-2)\hat{q}(w) - w\hat{q}(w) - 2w(w-2)\hat{q}'(w) = K$  yields  $(\ell-1)$  linear equations on the coefficients of  $\hat{q}$ , since the polynomial on the right hand side has degree not larger than  $\ell-1$ . Moreover, we can solve this equation uniquely by first defining the coefficient of  $w^{\ell-2}$  in  $\hat{q}$  with the help of the coefficient of  $w^{\ell-1}$  on both sides and then the lower order coefficients in the inverse order of their power. When we insert for  $\hat{q} = w^\ell \left(1 - \frac{2}{w}\right)^{-1/2}$ , then we have  $A = z^{2\ell+1}$ . This implies that  $\hat{q}$  is equal to the polynomial part of  $w^\ell \left(1 - \frac{2}{w}\right)^{-1/2}$ , and so indeed a solution.  $\square$

*Continuation of the proof of Proposition 9.5:* For this polynomial  $\widehat{q}$  we have

$$\begin{aligned} (2\ell+1)(w-2)\widehat{q}(w) - w\widehat{q}(w) - 2w(w-2)\widehat{q}'(w) &= \binom{-\frac{1}{2}}{\ell} (-2)^{\ell+1}(2\ell+1) \\ &= \frac{1.3\dots(2\ell-1)(2\ell+1)}{\ell!} (-2) \end{aligned}$$

The solution of the differential equation obeys at the initial value the equation

$$(9.12) \quad \dot{\alpha} = \frac{-(2\ell+1).\ell!}{2\alpha^\ell 1.3\dots(2\ell-1)(2\ell+1)} (C(0) + O(\alpha))$$

$$(9.13) \quad \dot{q}(w) = \frac{\dot{\alpha}}{2\alpha} (h_n(w) + O(\alpha))$$

where  $h_n(w)$  is the polynomial part of  $2w(q'(w) - \widehat{q}'(w)) - (2\ell+1)(q(w) - \widehat{q}(w)) + (1 - \frac{2}{w})^{-1}(q(w) - \widehat{q}(w))$  where  $\widehat{q}(w)$  is the polynomial in the lemma above. We remark that the first equation is a complex valued meromorphic equation. The second equation is the equation which involves the move of coefficients  $\gamma(t) - \widehat{\gamma} = \gamma(t) - \gamma(0)$ . To remove all poles of the vector field, we have to multiply the right hand side by  $\alpha^{\ell+1}$  such that  $\dot{q}$  does not have a pole. As a consequence  $\dot{\alpha}$  is equivalent to  $\alpha$  and  $\dot{q}$  is equivalent to  $q - \widehat{q}$ . So the linearized vector field has block diagonal form with respect to the decomposition of  $\alpha$  and  $q$ . Since we are interested in trajectories on which  $\dot{q}/\dot{\alpha}$  is bounded, we restrict to the eigenspaces of the  $\dot{\alpha}$  equation. For these eigenspaces  $\dot{\alpha}$  is not zero while  $\dot{q}$  vanishes in the lowest order.

It remains to find eigenvalues of the equation (9.12) with non zero real parts. We collect all equations corresponding to each  $m$ , and end up with equations of the form

$$\dot{\alpha}_m = \frac{C_m(1 + O(\alpha_1, \dots, \alpha_M))}{\alpha_m^{\ell_m}}$$

where  $\ell_m$  is the degree of  $q_m$  corresponding to  $A_m(z_m)$ , with  $m = 1, \dots, M$ . Let  $N$  be the least common multiple of  $\ell_1 + 1, \dots, \ell_M + 1$ . Then we use the parameters  $\alpha_1 = e^{i\theta_1} s^{\frac{N}{\ell_1+1}}$  and  $\alpha_m = e^{i\theta_m} r_m s^{\frac{N}{\ell_m+1}}$  for  $m > 1$  with real  $s, r_m, \theta_m$  to describe the evolution of  $A_1, \dots, A_M$  (The term  $O(\alpha_1, \dots, \alpha_M) = O(s^p)$  with  $p = \inf N/(\ell_m + 1)$ ). Here we assume that the root of  $b$  of index  $m = 1$  is a common root of  $a$  and  $b$ . Then with initial value  $(s_1(0), \theta_1(0)) = (0, \bar{\theta}_1)$  we have

$$\dot{\alpha}_1 = e^{i\theta_1} s^{\left(\frac{N}{\ell_1+1}-1\right)} \left( \frac{N}{\ell_1+1} \dot{s} + i s \dot{\theta}_1 \right).$$

Now we multiply the whole vectorfield with the real parameter  $s^N$ . The corresponding integral curves will be reparameterized by  $u = ts^N$ , but do not change as subsets in the space of polynomials  $(a, b)$ . Then we obtain for  $m = 1$  the equation

$$e^{i\theta_1} s^{\left(\frac{N}{\ell_1+1}-1\right)} \left( \frac{N}{\ell_1+1} \dot{s} + i s \dot{\theta}_1 \right) = \frac{C_1 s^{\frac{N}{\ell_1+1}}}{e^{i\ell_1 \theta_1}} (1 + O(s)).$$

This gives the system

$$\dot{\theta}_1 = \text{Im} \left( C_1 e^{-i(\ell_1+1)\theta_1} \right) + O(s^p), \quad \dot{s} = \frac{n+1}{N} s \text{Re} \left( C_1 e^{-i(\ell_1+1)\theta_1} \right) + O(s^{p+1}).$$

Now choose  $\bar{\theta}_1$  suitably to get  $\text{Im} \left( C_1 e^{-i(\ell_1+1)\bar{\theta}_1} \right) = 0$ , and recall that  $C_1 \neq 0$  by choosing the polynomial  $c$  without zeroes at branch points  $\alpha_1$ . This implies that

$$\frac{\partial}{\partial \theta_1} \text{Im} \left( C_1 e^{-i(\ell_1+1)\theta_1} \right) \Big|_{\theta_1=\bar{\theta}_1} = -(\ell_1+1) \text{Re} \left( C_1 e^{-i(\ell_1+1)\bar{\theta}_1} \right) = (\ell_1+1) \widehat{c}_1 \neq 0.$$

We have to choose the initial value  $\bar{\theta}_1$  at the starting point in the exceptional fibre of the blow up in such a way that  $\hat{c}_1 = \pm|C_1|$  has different sign. Then there exists different solution with negative and positive eigenvalues of the corresponding linearized equation

$$\dot{\theta}_1 = (\ell_1 + 1)\hat{c}_1\theta_1, \quad \dot{s} = \frac{n+1}{N}s \operatorname{Re} \left( C_1 e^{-(\ell_1+1)\bar{\theta}_1} \right) = -\frac{(\ell_1+1)\hat{c}_1}{N}s$$

For  $m > 1$ , we use parameters  $(s, r_m, \theta_m)$  with initial values  $(0, \bar{r}_m, \bar{\theta}_m)$  and  $\bar{r}_m \neq 0$ , for  $\alpha_m$  to get (with reparametrization by  $u = ts^N$ ) the equation

$$\begin{aligned} \dot{\alpha}_m &= \left( ir_m \dot{\theta}_m s + \dot{r}_m s + \frac{N}{\ell_m + 1} r_m \dot{s} \right) e^{i\theta_m s^{\frac{N}{n+1}-1}} \\ &= \frac{C_m s^N (1 + O(s^p))}{s^{\frac{\ell_m N}{\ell_m+1}} e^{i\ell_m \theta_m} r_m^{\ell_m}} = C_m e^{-i\ell_m \theta_m} r_m^{-\ell_m} s^{\frac{N}{\ell_m+1}} (1 + O(s^p)) \end{aligned}$$

This implies

$$ir_m \dot{\theta}_m s + \dot{r}_m s + \frac{N}{\ell_m + 1} r_m \dot{s} = C_m e^{-i(\ell_m+1)\theta_m} r_m^{-\ell_m} s (1 + O(s^p)).$$

Hence we study the system

$$\begin{aligned} \dot{\theta}_m &= \operatorname{Im} \left( C_m e^{-i(\ell_m+1)\theta_m} \right) r_m^{-(\ell_m+1)} + O(s^p) \\ \dot{r}_m &= \operatorname{Re} \left( C_m e^{-i(\ell_m+1)\theta_m} \right) r_m^{-\ell_m} - \frac{\ell_1 + 1}{\ell_m + 1} \operatorname{Re} \left( C_1 e^{-i(\ell_1+1)\bar{\theta}_1} \right) r_m + O(s^p) \end{aligned}$$

Now we choose  $\bar{\theta}_m$  in such a way that  $\operatorname{Im} \left( C_m e^{-i(\ell_m+1)\bar{\theta}_m} \right) = 0$  and if  $\hat{c}_m = -\operatorname{Re} \left( C_m e^{-i(\ell_m+1)\bar{\theta}_m} \right)$ , we can choose different values of  $\bar{\theta}_m$  to get different signs of  $\hat{c}_m = \pm|C_m|$ . For a choice of  $\bar{\theta}_1$ , and then a sign for  $\hat{c}_1$ , we fix a choice of  $(\bar{\theta}_2, \dots, \bar{\theta}_M)$  in such a way that  $\hat{c}_1$  and  $\hat{c}_m$  have same signs for  $m=2, \dots, M$ . We choose  $\bar{r}_m > 0$  which satisfy

$$\operatorname{Re} \left( C_m e^{-i(\ell_m+1)\bar{\theta}_m} \right) \bar{r}_m^{-(\ell_m+1)} - \frac{\ell_1 + 1}{\ell_m + 1} \operatorname{Re} \left( C_1 e^{-i(\ell_1+1)\bar{\theta}_1} \right) = 0$$

The linearized system is

$$\dot{\theta}_m = (\ell_m + 1)\hat{c}_m \bar{r}_m^{-(\ell_m+1)} \theta_m, \quad \dot{r}_m = K_m(\bar{\theta}, \bar{r}_m)(r_m - \bar{r}_m)$$

with  $K_m(\bar{\theta}, \bar{r}_m) = \frac{(\ell_m+1)}{\bar{r}_m^{\ell_m+1}} \hat{c}_m$ . Now there exists different choices of initial values, which fix the sign of eigenvalues for the linearized system. Thus we can find stable and unstable trajectories, and invoking the Stable Manifold Theorem concludes the proof.  $\square$

## 10. ISOLATED PROPERTY OF THE ABRESCH FAMILY

Consider an Abresch annulus  $X_1(\mathbb{C}/(\tau\mathbb{Z}) \times \mathbb{R})$  where  $\tau$  is the period of the cylinder in the  $x$ -direction. By proposition 3.5, the metric of  $X_1$  is induced by two elliptic functions  $x \rightarrow f(x) = \frac{-\omega_x}{\cosh \omega}$  and  $y \rightarrow g(y) = \frac{-\omega_y}{\cosh \omega}$ . The Jacobi operator on  $X_1$  is given by

$$\mathcal{L}_1 = \frac{1}{\cosh^2 \omega_1} \left( \partial_x^2 + \partial_y^2 + 1 + \frac{2|\nabla \omega_1|^2}{\cosh^2 \omega_1} \right) = \frac{1}{\cosh^2 \omega_1} (\partial_x^2 + \partial_y^2 + 1 + 2f^2(x) + 2g^2(y)).$$

We use Fourier analysis. We define the set of periodic eigenfunctions  $\{e_n\}$  associate to eigenvalues  $\lambda_0 < \lambda_1 \leq \lambda_2 \dots$  repeated with multiplicity,

$$(10.1) \quad \partial_x^2 e_n(x) + 2f^2(x)e_n(x) = -\lambda_n e_n(x).$$

where  $f$  is an elliptic function which satisfy

$$-(f_x)^2 = f^4 + (1 + c - d)f^2 + c = (f^2 - \delta_1)(f^2 - \delta_2).$$

If  $\tau$  denotes the period of the annulus in the  $x$ -direction, which coincides with a period of  $f$ , then the set  $\{e_n\}_{n \in \mathbb{N}}$  span the Hilbert space  $L^2(\mathbb{R} \setminus \tau\mathbb{Z})$ . A bounded solution of  $\mathcal{L}_1 v_0 = 0$  decomposes into

$$v_0(x, y) = \sum_{i \geq 0} u_n(y) e_n(x)$$

where  $u_n : \mathbb{R} \rightarrow \mathbb{R}$  are uniformly bounded functions.

**Lemma 10.1.** *Let  $u : \mathbb{C}/\tau\mathbb{Z} \rightarrow \mathbb{R}$  be a bounded solution of*

$$\mathcal{L}_1 u = \cosh^{-2} \omega_1 (\partial_x^2 u + \partial_y^2 u + u + 2f^2(x)u + 2g^2(y)u) = 0$$

where  $x \mapsto f(x)$  and  $y \mapsto g(y)$  are the functions defined in Theorem 3.5. Then  $u$  can not have more than 2 zeroes on horizontal sections unless it vanishes identically.

*Proof.* Consider elliptic equations which define functions  $x \mapsto f(x)$  and  $y \mapsto g(y)$

$$\begin{aligned} -(f_x)^2 &= f^4 + (1 + c - d)f^2 + c = (f^2 - \delta_1)(f^2 - \delta_2) \\ -(g_y)^2 &= g^4 + (1 + d - c)g^2 + d = (g^2 - \beta_1)(g^2 - \beta_2) \end{aligned}$$

with roots  $2\delta_1 = -(1 + c - d) + \sqrt{\Delta}$ ,  $2\delta_2 = -(1 + c - d) - \sqrt{\Delta}$ ,  $2\beta_1 = -(1 + d - c) + \sqrt{\Delta}$ ,  $2\beta_2 = -(1 + d - c) - \sqrt{\Delta}$  and  $\Delta = (1 + c - d)^2 - 4c = (1 + d - c)^2 - 4d$ , we see that  $f$  is oscillating around zero between  $-\sqrt{\delta_1}$  and  $\sqrt{\delta_1}$  since  $\delta_2 < 0$  and  $g$  is oscillating between  $-\sqrt{\beta_1}$  and  $\sqrt{\beta_1}$ .

We solve the equation (10.1) on  $[0, \tau/2]$ . The function  $\wp := \alpha - f^2$  (with  $3\alpha = -(1 + c - d)$ ) is the Weierstrass function which satisfies the elliptic equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

for some constant  $g_2, g_3$  depending only on constants  $c$  and  $d$ . The equation (10.1) transform into the Lamé equation

$$(10.2) \quad \partial_x^2 e_n - 2\wp e_n = -\mu_n e_n$$

On  $[0, \tau/2]$ , the functions  $e_0 = \sqrt{f^2 - \delta_2}$ ,  $e_1 = f$ ,  $e_2 = \sqrt{\delta_1 - f^2}$  are known as Lamé function of degree one of the first kind (see [26], chapter XXIII). These functions extend to  $[0, \tau]$  by symmetry and they are the first three eigenfunctions of the Lamé operator.

The function  $e_0 = \sqrt{f^2 - \delta_2} > 0$  is an eigenfunction associate to the eigenvalue  $\lambda_0 = -\delta_1$  with boundary data  $e_0(0) = e_0(\tau/2)$ ,  $\partial_x e_0(0) = \partial_x e_0(\tau/2) = 0$ . The second eigenfunction  $e_1 = f$  is associate to eigenvalue  $\lambda_1 = 1 + c - d$  with  $e_1(0) = -e_1(\tau/2)$ ,  $\partial_x e_0(0) = \partial_x e_0(\tau/2) = 0$ . The third function is  $e_2 = \sqrt{\delta_1 - f^2}$  associate to eigenvalue  $\lambda_2 = -\delta_2$  with  $e_2(0) = e_2(\tau/2) = 0$  and  $\partial_x e_0(0) = -\partial_x e_0(\tau/2)$ . These eigenfunctions extend by symmetry to  $\mathbb{R}/\tau\mathbb{Z}$ . They are the first three eigenfunctions of the spectrum with  $\lambda_0 < \lambda_1 < \lambda_2$  and have at most two zeroes on each horizontal curve. If  $e_k$  is an eigenfunction having strictly more than two zeroes on a period  $[0, \tau]$ , then the associated eigenvalues  $\lambda_k > \lambda_2$ . If not, one can argue by contradiction and compute  $W = e_k(\partial_x e_2) - (\partial_x e_k)e_2$ . Then  $W' = (\lambda_k - \lambda_2)e_k e_2$  and by studying the behavior of  $W$  between two consecutive zeroes of  $e_k$ , the function  $e_2$  has to change sign. Thus  $e_2$  would have at least four zeroes, a contradiction.

Now we consider  $u$  a bounded Jacobi field on  $\Sigma_0$ . By Fourier expansion we decompose  $u$  as

$$u(x, y) = \sum_{i \geq 0} u_n(y) e_n(x)$$

Since  $u$  is bounded on  $A_0$  then  $u_n$  is bounded on  $\mathbb{R}$ . Inserting  $u$  in the equation  $\mathcal{L}_0 u = 0$  we obtain a countable set of equations for  $n \in \mathbb{Z}$ :

$$\partial_y^2 u_n(y) + 2g^2(y)u_n(y) + (1 - \lambda_n)u_n(y) = 0.$$

For  $n \geq 2$ , we have  $(1 - \lambda_n) < 1 - \lambda_2 = 1 + \delta_2 = \frac{1}{2}(1 + d - c + \sqrt{\Delta}) = -\beta_1$ . But as we remarked for equation (10.1), the function  $\sqrt{g^2 - \beta_2} > 0$  is the first periodic eigenfunction associated with the first eigenvalue  $\mu_0 = -\beta_1$  of

$$(10.3) \quad \partial_y^2 v(y) + 2g^2(y) v(y) = -\mu v(y)$$

It is a well known fact (see [5] for example) that for  $\mu < \mu_0$  the equation (10.3) cannot have bounded solutions on  $\mathbb{R}$ . Then  $u_n = 0$  for  $n \geq 2$ . The function  $u$  is a linear combination of  $e_0, e_1, e_2$  and we obtain a contradiction with the following lemma.  $\square$

**Lemma 10.2.** *For any real constants  $\alpha_0, \alpha_1, \alpha_2$ , the function  $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2$  has at most two roots on  $\mathbb{R}/\tau\mathbb{Z}$ .*

*Proof.* The functions  $e_0, e_1$  and  $e_2$  obey  $e_0^2 + \delta_2 = e_1^2 = \delta_1 - e_2^2$ . Therefore the expression

$$(\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2)(\alpha_0 e_0 + \alpha_1 e_1 - \alpha_2 e_2)(\alpha_0 e_0 - \alpha_1 e_1 + \alpha_2 e_2)(\alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2)$$

is an even polynomial  $p(e_1)$  of degree four with respect to  $e_1$ . Along the period  $\tau$  the function  $e_1$  takes all values in  $(-\sqrt{\delta_1}, \sqrt{\delta_1})$  exactly twice. At two points in the preimage of one value of  $e_1$  the function  $e_0$  takes the same value and  $e_2$  takes values with opposite sign. Therefore at most two of the four roots of  $p(e_1) = 0$  are the values of  $e_1 \in [-\sqrt{\delta_1}, \sqrt{\delta_1}]$  at a root of  $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2$ . The negative of these roots are the values of  $e_1$  at roots of  $\alpha_0 e_0 - \alpha_1 e_1 + \alpha_2 e_2$ .  $\square$

## 11. BUBBLETONS ON ABRESCH FAMILY ARE NOT EMBEDDED

**Bubbletons on flat annuli.** We consider the spectral data of the flat cylinder and we prove that a flat cylinder with bubbletons cannot be embedded.

**Proposition 11.1.** *There are no embedded bubbletons of spectral genus zero.*

*Proof.* The flat cylinder has a spectral curve of genus zero. Assume that there is an embedded bubbleton dressed at  $\alpha_0$  with  $|\alpha_0| \neq 1$ . Since the bubbleton is embedded, removing the bubbleton gives the embedded flat cylinder, then the period of the bubbleton is the same as the period of the flat cylinder. The length of this period is  $|\tau| = 2\pi$ . The bubbleton occurs at a double point  $\alpha_0$  where  $\mu(\alpha_0)^2 = 1$  and  $\mu = f\nu + g$  with  $f(\alpha_0) = 0$ . We add to  $a(\lambda)$  a double zero and to  $b(\lambda)$  a simple zero at  $\alpha_0$  and  $\bar{\alpha}_0^{-1}$ , and we apply the parametrization of section 9.2 to obtain

$$\tilde{a}(\lambda) = (1 - \lambda\bar{\alpha}_0)^2(\lambda - \alpha_0)^2 a(\lambda) \text{ and } \tilde{b}(\lambda) = (1 - \lambda\bar{\alpha}_0)(\lambda - \alpha_0)b(\lambda).$$

Since the bubbleton is assumed to be embedded we can open the spectral curve at the double point  $\alpha_0$ , increasing strictly the length of  $|\tau|$  in the family of genus 2 spectral curves by choosing  $\tilde{c}(\tilde{\lambda})$  as in proposition 9.3 and remark 9.4. Along this deformation the annulus stays embedded. Now we increase the period  $|\tau|$  up to its maximum. At this maximum we cannot find enough degrees of freedom in  $c$  and so we are in a spectral genus zero example, closed along a period  $|\tau| > 2\pi$ . This flat annulus is a covering of the embedded flat annulus since the length of the horizontal geodesic are larger than  $2\pi$ . Since the deformation we use preserves embeddedness, the bubbleton cannot be embedded.  $\square$

**Bubbleton on Riemann's type example.** We consider the spectral data  $(\tilde{a}, \tilde{b})$  of an Abresch annulus. This data satisfies (see appendix)

$$\lambda^{2g} \overline{\tilde{a}(1/\bar{\lambda})} = \tilde{a}(\lambda) \text{ and } \lambda^{g+1} \overline{\tilde{b}(1/\bar{\lambda})} = -\tilde{b}(\lambda)$$

- a)  $\lambda^{2g} \tilde{a}(1/\lambda) = \tilde{a}(\lambda)$  and  $\lambda^{g+1} \tilde{b}(1/\lambda) = \tilde{b}(\lambda)$  if  $\tilde{a}$  has a root  $\alpha \in \mathbb{R}^+$  and  $\tilde{b}(0) \in i\mathbb{R}$
- b)  $\lambda^{2g} \tilde{a}(1/\lambda) = \tilde{a}(\lambda)$  and  $\lambda^{g+1} \tilde{b}(1/\lambda) = -\tilde{b}(\lambda)$  if  $\tilde{a}$  has only roots in  $\mathbb{R}^-$  and  $\tilde{b}(0) \in \mathbb{R}$

Now consider a bubbleton on  $(\tilde{a}, \tilde{b})$ . Bubbletons occur at  $\alpha_0$  away from  $\mathbb{S}^1$ . The spectral curve of a simple bubbleton is singular and

$$\Sigma = \{(\nu, \lambda) \in \mathbb{C}^2; \nu^2 = \lambda^{-1}a(\lambda) = \lambda^{-1}(\lambda - \alpha_0)^2(1 - \lambda\alpha_0)^2\tilde{a}(\lambda)\}$$

To close the period, we need  $\tilde{F}_{\alpha_0}(\tau) = \pm \text{Id}$ , where  $\tau$  is a period of the annulus  $(\tilde{a}, \tilde{b})$ . The set of bubbletons at  $\alpha_0$  is described by an orbit homeomorphic to  $\mathbb{CP}^1$ . If the bubbleton closes with period  $\tau$ , then there is a holomorphic map  $\mu = f\nu + g$  on  $\Sigma - \{0, \infty\}$ . The map  $\nu$  has simple roots at  $\alpha_0$  and  $1/\bar{\alpha}_0$  and  $\nu = (\lambda - \alpha_0)(1 - \lambda\alpha_0)\tilde{\nu}$  where  $\nu^2 = \lambda^{-1}\tilde{a}(\lambda)$ . We see that  $\mu = f\nu + g = (\lambda - \alpha_0)(1 - \lambda\alpha_0)f\tilde{\nu} + g$  is holomorphic on  $\tilde{\Sigma} - \{0, \infty\}$  and  $\mu(\alpha_0) = g(\alpha_0) = \pm 1$  (by hyperelliptic involution).

This means that if there is a periodic bubbleton with period  $\tau$ , then we have an isospectral set of periodic bubbletons with the same period. In this isospectral orbit there is a potential  $\xi_\lambda$  with a root at  $\alpha_0$ . This annulus has a removable singularity and we see that  $\mu$  defines a holomorphic periodic map, and the same period  $\tau$  closes the annulus defined by the potential  $\xi_\lambda$ . In summary, bubbletons occur at double points of  $\mu = \tilde{f}\tilde{\nu} + \tilde{g}$ , that is where  $\mu(\alpha_0) = \pm 1$ .

**Theorem 11.2.** *There is no embedded bubbleton in the family of Abresch annuli.*

*Proof.* We assume the existence of an embedded bubbleton. This bubbleton has  $2n$  higher order roots. We can apply the isospectral action to inductively reduce the order of the bubbleton. To do that assume the potential  $\xi_\lambda = p(\lambda)h_{L', \alpha_0}\tilde{\xi}_\lambda h_{L', \alpha_0}^{-1}$  induces the bubbleton. Then we can find an isospectral action, which changes  $L'$  and preserve  $\tilde{\xi}_\lambda$ . At the closure of the limit we produce a potential with removable singularity at  $\alpha_0$ . If the bubbleton is embedded, then we obtain an embedded bubbleton with  $2n - 2$  higher order roots. Inductively we find a bubbleton on an Abresch annulus with a double root  $\alpha_0$  of order 2.

The point  $\alpha_0$  is not a root of  $\tilde{a}(\lambda)$ , because if it were, then would have  $d\mu(\alpha_0) = 0$  and then  $\alpha_0$  would be a common root of  $\tilde{a}$  and  $\tilde{b}$ . But  $\tilde{b}$  never has roots at the roots of  $\tilde{a}$  by Proposition C.1. Thus bubbletons occur only away from  $\mathbb{S}^1$ , and away from the roots of  $\tilde{a}(\lambda)$ . A study of spectral data in Lemma C.2 show that at roots of  $\tilde{b}$  the function  $\mu$  cannot have double points. Thus double points do not occur at roots of  $\tilde{a}$  and roots of  $\tilde{b}$ .

This property holds for every spectral genus 0, 1 or 2 spectral curve describing an Abresch annulus. Now we can apply deformation of Theorem 9.2 where  $\tilde{a}$  and  $\tilde{b}$  have no common roots, increasing the flux in order to reduce the spectral genus and end with spectral data  $(\tilde{a}, \tilde{b})$  of the flat annulus. We deform the bubbleton on the Abresch annulus (of spectral genus 1 or 2) into a bubbleton on a flat cylinder. It suffices to keep the period closed of the bubbleton along the deformation so that  $\tilde{F}_{\alpha_0}(\tau) = \pm \text{Id}$  where  $\alpha_0$  is a double point. Here double points of  $\mu$  evolve by the equation (see proof of theorem 9.2)

$$\frac{\dot{\alpha}_0}{\alpha_0} = -\frac{\tilde{c}(\alpha_0)}{\tilde{b}(\alpha_0)}.$$

Since  $\tilde{b}(\alpha_0) \neq 0$ , the equation is well defined. The double point moves along the trajectory of spectral data and the bubbleton remains closed and embedded by proposition 2.8. We can follow the double roots along the deformation of Abresch annuli, increasing the flux and ending at the flat annulus.

To prove that at the end we have a bubbleton on a flat cylinder, there remains to prove that the double point  $\alpha_0$  is not converging to  $\mathbb{S}^1$  along the deformation (if not the double roots can be removed without changing the geometry of the annulus).

By symmetry  $\mu(\bar{\alpha}_0^{-1}) = \pm 1$  and if during the deformation the point  $\alpha_0$  is going to  $\mathbb{S}^1$ , then the two double points  $\alpha_0$  and  $\bar{\alpha}_0^{-1}$  must coalesce. Then we have a root of  $d \ln \mu$  and thus a zero of



$\tilde{b}$  on  $\mathbb{S}^1$ . This can happen only at  $\lambda = -1$  or  $\lambda = 1$  by proposition C.1, where zeroes of  $\tilde{b}$  are given.

Now we study zeroes of the function  $\mu^2 - 1$  on the spectral curve  $\Sigma$ . For a flat annulus we have two roots of  $\mu^2 - 1$ . Since  $\ln \mu = \frac{i\pi}{2}(\sqrt{\lambda} + 1/\sqrt{\lambda})$ , we have two roots of order two at  $\lambda = 1$  and two roots of order one at  $\lambda = -1$  (by hyperelliptic involution we have roots on  $(\lambda_0, \nu)$  and  $\lambda_0, -\nu$ ) when  $\lambda_0$  is not a zero of  $a(\lambda)$ .

For genus one with  $0 < \alpha < 1$ , we have two roots of order one at  $\lambda = -1$  and two roots of order one at the branch points  $\alpha, \bar{\alpha}^{-1}$ , and two roots of order one at  $\lambda = 1$ .

For genus one with  $-1 < \alpha < 0$ , we have two roots of order two at  $\lambda = 1$  and two roots of order one at the branch points and no roots at  $\lambda = -1$ .

For genus two, we have one root of order one at each branch point  $\alpha, \beta, \bar{\alpha}^{-1}, \bar{\beta}^{-1}$ , two roots of order one at  $\lambda = 1$  and no roots at  $\lambda = -1$ .

Beside these roots there are no additional roots of  $\mu^2 - 1$  nearby  $\lambda = \pm 1$ . This means that along a deformation converging to the flat annulus, the roots of  $\mu^2 - 1$  in  $\mathbb{C}^* - \mathbb{S}^1$  away from the branch points cannot converge to  $\mathbb{S}^1$ . The roots which sit at  $\lambda = \pm 1$  are limits of roots listed above and thus there is no additional one, since  $\alpha_0$  is not amongst the branch points of the spectral curve.

Along this deformation the roots of  $\mu^2 - 1$  cannot go to infinity because zeroes of  $b$  are bounded. We remark that the equation  $\frac{\dot{\alpha}_0}{\alpha_0} = -\frac{c(\alpha_0)}{b(\alpha_0)}$  cannot provide a solution where  $\alpha_0(t) \rightarrow \infty$  in finite time, since the vector field  $c$  is polynomial in  $\alpha_0$ , and  $b(\alpha_0)$  is bounded away from zero when  $\alpha_0$  goes to infinity.

Hence our deformation deforms a bubbleton on a Riemann type example to a bubbleton on a flat cylinder. Since the deformation preserves embeddness, this yields an embedded bubbleton on the flat cylinder, a contradiction.  $\square$

## APPENDIX A. SYM POINT

**Proposition A.1.** *Let  $\zeta_\lambda : \mathbb{R}^2 \rightarrow \mathcal{P}_g$  be a polynomial Killing field (6.3) with initial condition  $\xi_\lambda \in \mathcal{P}_g$ . Let  $F_\lambda$  be the corresponding extended frame  $F_\lambda^{-1}dF_\lambda = \alpha(\zeta_\lambda)$  and the immersion  $X_{\lambda_0}(z) = (F_{\lambda_0}(z)\sigma_3 F_{\lambda_0}^{-1}(z), \text{Re}(-iz))$  is parameterized by its third coordinate with  $\lambda_0 = e^{i\theta}$ . The Hopf differential is given by  $Q_{\lambda_0}(z) = -4\beta_{-1}\gamma_0\lambda_0^{-1}(dz)^2 = \frac{1}{4}(dz)^2$ . Then the map  $\tilde{F}_\lambda(z) = F_{e^{i\theta}\lambda}(e^{i(1-g)\theta/2}z)$  is the unitary factor of the initial value*

$$\tilde{\xi}_\lambda = e^{i(1-g)\theta/2}\xi_{e^{i\theta}\lambda}.$$

*In particular we have*

$$\det \tilde{\zeta}_\lambda(z) = \det \tilde{\xi}_\lambda = -\lambda^{-1}\tilde{a}(\lambda) = -\lambda^{-1}e^{-ig\theta}a(e^{i\theta}\lambda).$$

*The immersion is locally given by*

$$\tilde{X}_1(z) = X_{\lambda_0}(e^{i(1-g)\theta/2}z) = (\tilde{F}_1(z)\sigma_3\tilde{F}_1^{-1}(z), \text{Re}(-ie^{i(1-g)\theta/2}z)).$$

*Proof.* The matrix  $\tilde{\xi}_\lambda = \begin{pmatrix} \tilde{\alpha}(\lambda) & \tilde{\beta}(\lambda) \\ \tilde{\gamma}(\lambda) & -\tilde{\alpha}(\lambda) \end{pmatrix} \in \mathcal{P}_g$  satisfies the reality condition with

$$\tilde{\alpha}(\lambda) = e^{i(1-g)\theta/2}\alpha(e^{i\theta}\lambda), \quad \tilde{\beta}(\lambda) = e^{i(1-g)\theta/2}\beta(e^{i\theta}\lambda) \quad \text{and} \quad \tilde{\gamma}(\lambda) = e^{i(1-g)\theta/2}\gamma(e^{i\theta}\lambda).$$

The matrix  $\tilde{\xi}_\lambda$  has unitary factor in Iwasawa decomposition  $\tilde{F}_\lambda(z) = F_{e^{i\theta}\lambda}(e^{i(1-g)\theta/2}z)$ . The Hopf differential associate to  $\tilde{F}_1(z)$  is given by

$$\tilde{Q}_1 = -4\tilde{\beta}_{-1}\tilde{\gamma}_0(dz)^2 = -4\tilde{a}(0)(dz)^2 = -4e^{-ig\theta}\beta_{-1}\gamma_0(dz)^2 = \frac{1}{4}e^{i(1-g)\theta}(dz)^2.$$

which proves the formula for  $\tilde{X}_1$ .  $\square$

## APPENDIX B. BUBBLETONS

Bubbletons occur when  $a(\lambda) = (\lambda - \alpha_0)^2(1 - \lambda\bar{\alpha}_0)^2\tilde{a}(\lambda)$  has roots of higher order, and then the spectral curve  $\Sigma$  is singular. The idea is to construct a group action which deforms an embedded bubbleton associated to  $\xi_\lambda \in I(a)$  into an embedded annulus induced by a potential  $\tilde{\xi}_\lambda \in I(\tilde{a})$ . Inductively we can remove all roots of higher order of  $a(\lambda)$ .

If  $\xi_{\alpha_0} \neq 0$  and  $\det \xi_{\alpha_0} = 0$ , then the matrix is nilpotent and defines a complex line  $L = \ker \xi_{\alpha_0} = \text{Im } \xi_{\alpha_0} \in \mathbb{CP}^1$ . This complex line  $L$  determines a simple factor having a pole at  $\alpha_0$  and  $\bar{\alpha}_0^{-1}$ . Technical details on simple factor dressing are given in section 5 of [8]. Roughly speaking, the idea is to consider the following matrix

$$\pi_{\alpha_0}(\lambda) := \begin{pmatrix} \sqrt{\frac{\lambda - \alpha_0}{1 - \bar{\alpha}_0 \lambda}} & 0 \\ 0 & \sqrt{\frac{1 - \bar{\alpha}_0 \lambda}{\lambda - \alpha_0}} \end{pmatrix}$$

and  $\pi_{L, \alpha_0} = Q_L \pi_{\alpha_0} Q_L^{-1}$ , where  $L, L^\perp$  are eigenlines of  $\pi_{L, \alpha_0}$  by changing the basis of  $\mathbb{C}^2$  ( $Q_L \in \text{SU}_2$  and  $\langle Q_L e_1 \rangle = L$ ). To get a simple factor, we need to find the correct matrix  $Q_{1, L'}$  in  $\text{SU}_2$  (by Gram-Schmidt orthogonalization) depending only on  $L'$  in order to satisfy at  $\lambda = 0$

$$h_{L', \alpha_0}(0) = Q_{1, L'} \pi_{L', \alpha_0}^{-1}(0) \in \Lambda_r^+ \text{SL}_2$$

with  $r < |\alpha_0| < 1$ . This means that at  $\lambda = 0$  the matrix  $h_{L', \alpha_0}(0)$  is an upper triangular matrix. Moreover, if  $L$  is given, there is a unique  $L'$  depending only on  $L$  such that  $h_{L', \alpha_0}(\lambda)(L') = L$  (see Lemma 5.1, section 5, [8]). We denote this change by  $L' = \hat{Q}_L L$  for some  $\hat{Q}_L \in \text{SU}_2$  (see corollary 5.1 [8]).

Any potential  $\xi_\lambda \in I(a)$  decomposes uniquely into an element  $(L', \tilde{\xi}_\lambda) \in \mathbb{CP}^1 \times I(\tilde{a})$  by

$$\xi_\lambda = p(\lambda) h_{L', \alpha_0} \tilde{\xi}_\lambda h_{L', \alpha_0}^{-1}$$

where  $p(\lambda) = (\lambda - \alpha_0)(1 - \bar{\alpha}_0 \lambda)$  and  $L' = \hat{Q}_L L$  with  $L = \text{Ker } \xi_{\alpha_0}$ . In the case where  $(L'_0)^\perp$  is an eigenline of  $\tilde{\xi}_{\alpha_0}$ , the potential  $\xi_\lambda$  has a zero at  $\alpha_0$  and we can remove the zero without changing its extended frame  $F_\lambda$ .

For any potential  $\xi_\lambda$  which decomposes into  $(L', \tilde{\xi}_\lambda)$  we consider the change  $(L'(t), \tilde{\xi}_\lambda)$  and change  $L'(t)$  to  $L'_1$  where  $(L'_1)^\perp$  is an eigenline of  $\tilde{\xi}_{\alpha_0}$ . This change preserves embeddedness and periodicity as it is isospectral. The isospectral action acts transitively on the first factor  $L' \in \mathbb{CP}^\times = \mathbb{CP}^1 - \{L'_1, L'_2\}$  where  $(L'_1)^\perp, (L'_2)^\perp$  are eigenlines of  $\tilde{\xi}_{\alpha_0}$  (eventually there is only one eigenline  $(L'_1)^\perp$ ).

We consider the unitary factor  $F_\lambda : \mathbb{R}^2 \rightarrow \Lambda_r \text{SU}_2(\mathbb{C})$  of the r-Iwasawa decomposition

$$\exp(z\xi_\lambda) = F_\lambda B_\lambda$$

and we define  $\tilde{F}_\lambda : \mathbb{R}^2 \rightarrow \Lambda \text{SU}_2(\mathbb{C})$  the unitary factor of the r-Iwasawa decomposition

$$\exp(zp(\lambda)\tilde{\xi}_\lambda) = \tilde{F}_\lambda \tilde{B}_\lambda.$$

Terng-Uhlenbeck [24] provide the following useful formula.

**Proposition B.1.** *Let  $h_{L', \alpha_0}$  the simple factor with  $\alpha_0 \in \mathbb{C}$ ,  $r < |\alpha_0| < 1$  and  $L \in \mathbb{CP}^1$ . Then*

$$F_\lambda(z) = h_{L', \alpha_0} \tilde{F}_\lambda(z) h_{L'(z), \alpha_0}^{-1} \text{ with } L'(z) = {}^t \bar{\tilde{F}}_{\alpha_0}(z) L'$$

It turns out that changing the line  $L'$  is isospectral and preserves the closing conditions. To investigate this restricted isospectral action we have the following

**Theorem B.2.** [8] *We consider a decomposition of a potential  $\xi_\lambda \in I(a)$  in  $(L', \tilde{\xi}_\lambda) \in \mathbb{CP}^1 \times I(\tilde{a})$  and the two-dimensional subgroup action of  $\mathbb{R}^{2g}$ ,  $\tilde{\pi}(\cdot)\xi_\lambda : \mathbb{C} \rightarrow I(a)$  given by*

$$\tilde{\pi}(\beta)\xi_\lambda = \begin{cases} \left( \frac{\beta}{\lambda - \alpha_0} + \frac{\bar{\beta}\lambda}{1 - \bar{\alpha}_0\lambda} \right) \lambda^{\frac{1-g}{2}} \xi_\lambda & \text{when } g = 2k + 1 \\ \left( \frac{\beta}{\lambda - \alpha_0} + \frac{\bar{\beta}\lambda}{1 - \bar{\alpha}_0\lambda} \right) (\lambda^{\frac{-g}{2}} + \lambda^{1-\frac{g}{2}}) \xi_\lambda & \text{when } g = 2k. \end{cases}$$

*This subgroup acts on  $\xi_\lambda = (L', \tilde{\xi}_\lambda)$  by preserving the second term  $\tilde{\xi}_\lambda$  of the decomposition. If we denote by  $\tilde{\pi}(\beta)\xi_\lambda = (L'(\beta), \tilde{\xi}_\lambda)$ , the subgroup  $\mathbb{C}$  acts on  $L' \in \mathbb{CP}^1 \setminus \{L'_1, L'_2\}$  transitively where  $(L'_1)^\perp, (L'_2)^\perp$  are eigenlines of  $\tilde{\xi}_{\alpha_0}$  and fixed point of the action.*

We show that the action acts transitively on the first factor and preserves the second factor  $\tilde{\xi}_\lambda$  to prove that along this deformation embeddedness and closing conditions are preserved.

**Proposition B.3.** [8] *If there is  $(L'_1, \tilde{\xi}_\lambda) \in \mathbb{CP}^1 \times I(\tilde{a})$ , such that  $\xi_{1,\lambda} = p(\lambda)h_{L'_1, \alpha_0} \tilde{\xi}_\lambda h_{L'_1, \alpha_0}^{-1}$  induces an embedded minimal annulus with period  $\tau$ , then for any  $L'_2 \in \mathbb{CP}^1$ , the potential  $\xi_{2,\lambda} = p(\lambda)h_{L'_2, \alpha_0} \tilde{\xi}_\lambda h_{L'_2, \alpha_0}^{-1}$  yields an embedded minimal annulus with the same period  $\tau$ .*

**Corollary B.4.** [8] *This group action defines an orbit which preserves the degree of the roots of the potential  $\xi_\lambda$ . If  $a(\lambda) = (\lambda - \alpha_0)^2(1 - \lambda\bar{\alpha}_0)^2\tilde{a}(\lambda)$ , then the closure of the set  $N = \{\xi_\lambda \in I(a) \mid \xi_{\alpha_0} \neq 0\}$  is isomorphic to the set  $(\mathbb{CP}^1) \times I(\tilde{a}) = \bar{N}$ .*

## APPENDIX C. ABRESCH ANNULI

### C.1. Spectral data.

**Proposition C.1.** *The spectral curve of genus 0 associate to an embedded annulus parameterized by its Sym point at  $\lambda = 1$  is given by*

$$1) \ a(\lambda) = \frac{-1}{16} \text{ and } b(\lambda) = \frac{\pi}{16}(\lambda - 1)$$

*The spectral curve of genus 1 associate to an embedded annulus parameterized by its Sym point at  $\lambda = 1$  is given by*

$$2) \ a(\lambda) = \frac{1}{16\alpha}(\lambda - \alpha)(\alpha\lambda - 1) \text{ for } \alpha \in (0, 1) \text{ and } b(\lambda) = \frac{b(0)}{\gamma}(\lambda - \gamma)(\gamma\lambda - 1) \text{ with } \gamma \in (\alpha, 1) \text{ and } b(0) \in \mathbb{i}\mathbb{R} \text{ both determined by } \alpha.$$

$$3) \ a(\lambda) = \frac{-1}{16\beta}(\lambda + \beta)(\beta\lambda + 1) \text{ for } \beta \in (0, 1) \text{ and } b(\lambda) = \frac{b(0)}{\gamma}(1 - \lambda)(1 + \lambda) \text{ and } b(0) \in \mathbb{R} \text{ determined by } \beta.$$

*The spectral curve of genus 2 associate to an embedded annulus parameterized by its Sym point at  $\lambda = 1$  is given by*

$$4) \ a(\lambda) = \frac{1}{16\beta\alpha}(\lambda - \alpha)(\alpha\lambda - 1)(\lambda + \beta)(\beta\lambda + 1) \text{ for } \alpha, \beta \in (0, 1) \text{ and } b(\lambda) = \frac{b(0)}{\gamma}(1 + \lambda)(\lambda - \gamma)(\gamma\lambda - 1) \text{ for } \gamma \in (\alpha, 1) \text{ and } b(0) \in \mathbb{i}\mathbb{R} \text{ both determined by } \alpha \text{ and } \beta.$$

*In conclusion, the polynomial  $a$  satisfies the additional symmetry  $\lambda^{2g}a(1/\lambda) = a(\lambda)$  and*

- a)  $\lambda^{g+1}b(1/\lambda) = b(\lambda)$  if  $a$  has a root  $\alpha \in \mathbb{R}^+$  and  $b(0) \in \mathbb{i}\mathbb{R}$
- b)  $\lambda^{g+1}b(1/\lambda) = -b(\lambda)$  if  $a$  has only roots in  $\mathbb{R}^-$  and  $b(0) \in \mathbb{R}$

*Proof.* The proof is given in [8]. The Abresch system of proposition 3.5 gives the relation  $\omega_{zzzz} - 2\omega_z^3 = -\frac{1}{4}\omega_{\bar{z}} + \frac{c-d}{2}\omega_z$ . We apply the iteration of Pinkall-Sterling described in proposition 5.2 and obtain a corresponding polynomial  $a(\lambda)$ . Finally we prove in [8] that the polynomial  $b$  satisfies the closing conditions.  $\square$

**Lemma C.2.** *If  $\gamma$  is a root of  $b$  then the corresponding function  $|\mu(\gamma)| \neq 1$ .*

*Proof.* For  $\lambda \in [\alpha, \bar{\alpha}^{-1}]$ , the function  $h = \ln \mu$  is real and

$$\int_{\alpha}^{1/\alpha} \frac{b d\lambda}{\nu \lambda^2} = 2 \int_{\alpha}^1 \frac{b d\lambda}{\nu \lambda^2} = \int_{\alpha}^1 dh = 0.$$

Then  $\gamma$  is a root of  $dh$  and is contained in the interval  $(\alpha, 1)$ . Since  $\operatorname{Re} h(\alpha) = \operatorname{Re} h(1) = 0$ , the value  $\gamma$  is the local critical point of  $h$  and then  $\operatorname{Re} h(\gamma) \neq 0$ , so  $|\mu| \neq 1$ .  $\square$

**C.2. Whitham deformation of spectral genus 0, 1 and 2.** We apply the deformation in the case where the spectral curve has spectral genus one or two. We show that the family of Riemann type examples is a two parameter family.

The space of polynomials  $a$  of degree  $2g$  which obey  $\lambda^{2g} \overline{a(\bar{\lambda}^{-1})} = a(\lambda)$  is a real  $2g+1$  dimensional vector space with  $a_g \in \mathbb{R}, a_0, \dots, a_{g-1} \in \mathbb{C}$  and  $(a_{2g}, \dots, a_{g+1}) = (\bar{a}_0, \dots, \bar{a}_{g-1})$ . The value  $|a_0|$  is independent of the roots of  $a$ . All other  $2g$  degrees of freedom are determined by the roots of  $a$ . The space of polynomials  $b$  or  $c$  is a real  $g+2$  dimensional vector space. Then the set of spectral data  $(a, b)$  has  $(3g+3)$ -degrees of freedom. By choosing a polynomial  $c$ , we have

$$(C.1) \quad -2\dot{b}a + b\dot{a} = -2\lambda ac' + ac + \lambda a'c = p(\lambda)$$

which is a set of  $(3g+2)$ -real equations (since  $\lambda^{3g+1} \overline{p(1/\bar{\lambda})} = -p(\lambda)$ ). These equations determine  $(\dot{a}, \dot{b})$  in terms of  $c$ . To preserve the closing condition we need  $c(1) = 0$  and one real condition  $\operatorname{Im} \frac{c(0)}{b(0)} = 0$ .

We consider the embedded flat cylinder as described in Proposition C.1. The spectral data are given by  $a_0(\lambda) = -1/16$  and  $b_0(\lambda) = \frac{\pi}{16}(\lambda - 1)$ . We look for all deformations of  $(a_0, b_0)$  preserving embeddedness in the family of finite type annuli of genus 0, 1 or 2. We determine the values of  $\mu = \pm 1$  on the unit circle. The point at  $\lambda = +1$  and  $\lambda = -1$  are available and we can open nodes of the spectral curve preserving the closing condition and embeddedness.

Therefore we can deform  $a(\lambda) = \frac{1}{16}(\lambda - 1)^2(\lambda + 1)^2$  and  $b(\lambda) = \frac{\pi i}{16}(\lambda - 1)^2(\lambda + 1)$ . The corresponding  $c$ 's have to obey  $c(1) = 0$  and  $\operatorname{Re} c(0) = 0$ . The solution space is the two-dimensional space spanned by  $i(\lambda^3 - 1)$  and  $i(\lambda^2 - \lambda)$ . Therefore they obey

$$\lambda^3 c(1/\lambda) = -c(\lambda).$$

This implies that all of them preserve the symmetry

$$\lambda^4 a(1/\lambda) = a(\lambda) \quad \lambda^3 b(1/\lambda) = b(\lambda) \quad \lambda^3 c(1/\lambda) = -c(\lambda).$$

The solution is calculated in [8, Proposition 6.3]. It is a two-dimensional family parameterized by  $(\alpha, \beta) \in (0, 1] \times (0, 1]$

$$a(\lambda) = \frac{1}{\beta\alpha}(\lambda - \alpha)(\alpha\lambda - 1)(\lambda + \beta)(\beta\lambda + 1) \quad b(\lambda) = \frac{b(0)}{\gamma}(1 + \lambda)(\lambda - \gamma)(\gamma\lambda - 1)$$

with  $b(0) \in i\mathbb{R}$  and  $\gamma \in [\alpha, 1]$  determined by  $\alpha$  and  $\beta$ . The unique genus zero example corresponds to  $\alpha = 1 = \beta$  and the two genus one families to  $\alpha = 1$  and  $\beta \in (0, 1)$  and  $\alpha \in (0, 1)$  and  $\beta = 1$ . In these cases double roots of  $a$  can be cancelled with simple roots of  $b$ . Due to the normalization, in the first case together with the cancellation  $a$  is multiplied with  $-1$  and  $b$  with  $i$  and the action of the symmetry changes. This is a two parameter family of embedded annuli, describing the space moduli of Riemann type examples.

**Remark C.3.** *If there were double points on the unit circle not situated at  $\lambda = 1$  or  $\lambda = -1$ , then these could be used to deform an Abresch annulus into a higher spectral genus annulus. In an other way this is impossible by uniqueness of Abresch annuli. This would produce a Jacobi field with four zeroes on each horizontal section. This argument proves that the only double points are at  $\lambda = 1$  and  $\lambda = -1$ . An alternative way to prove the isolated property is to determine exactly the set of double point on the unit circle.*

## REFERENCES

- [1] U. Abresch, *Constant mean curvature tori in terms of elliptic functions*, J. Reine U. Angew Math. **374** (1987), 169–192.
- [2] A. I. Bobenko, *All constant mean curvature tori in  $\mathbb{R}^3$ ,  $S^3$ ,  $\mathbb{H}^3$  in terms of theta-functions*, Math. Ann. **290** (1991), 209–245.
- [3] S.-Y. Cheng, *Eigenfunctions and nodal sets*, Comment. Math. Helv. **51** (1976), no. 1, 43–55.
- [4] R. Sa Earp and E. Toubiana, *Screw motion surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  and  $S^2 \times \mathbb{R}$* , Illinois J. Math. **49** (2005), no. 4, 1323–1362.
- [5] M. S. P. Eastham, *Results and problems in the spectral theory of periodic differential equations*, Springer, Berlin, 1975.
- [6] O. Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, vol. 81, Springer-Verlag, New York, 1991.
- [7] L. Hauswirth, *Minimal surfaces of Riemann type in three-dimensional product manifolds*, Pacific J. Math. **224** (2006), no. 1, 91–117.
- [8] L. Hauswirth, M. Kilian, and M. U. Schmidt, *Finite type minimal annuli in  $S^2 \times \mathbb{R}$* , preprint, 2012.
- [9] ———, *On the moduli of constant mean curvature annuli of finite type in the 3-sphere*, preprint, 2012.
- [10] E. Heintze and H. Karcher, *A general comparison theorem with applications to volume estimates for submanifolds*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 451–470.
- [11] D. Hoffman and B. White, *Axial minimal surfaces in  $S^2 \times \mathbb{R}$  are helicoidal*, J. Differential Geom. **87** (2011), no. 3, 515–523.
- [12] S. B. Jackson, *The four-vertex theorem for surfaces of constant curvature*, Amer. J. Math. **67** (1945), 563–582.
- [13] H. Karcher, *Riemannian comparison constructions*, Global differential geometry, MAA Stud. Math., vol. 27, Math. Assoc. America, Washington, DC, 1989, pp. 170–222.
- [14] R. B. Lockhart and R. C. McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 3, 409–447.
- [15] L. Mazet, *A general halfspace theorem for constant mean curvature surfaces*, American Journal Math **137** (2009), no. 8, 2761–2765.
- [16] I. McIntosh, *Harmonic tori and their spectral data*, Surveys on geometry and integrable systems, Adv. Stud. Pure Math., vol. 51, Math. Soc. Japan, Tokyo, 2008, pp. 285–314.
- [17] W. H. Meeks III, J. Pérez, and A. Ros, *Properly embedded minimal planar domain*, in preparation.
- [18] W. H. Meeks III and H. Rosenberg, *Stable minimal surfaces in  $M \times \mathbb{R}$* , J. Differential Geom. **68** (2004), no. 3, 515–534.
- [19] ———, *The theory of minimal surfaces in  $M \times \mathbb{R}$* , Comment. Math. Helv. **80** (2005), no. 4, 811–858.
- [20] R. H. L. Pedrosa and M. Ritoré, *Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems*, Indiana Univ. Math. J. **48** (1999), no. 4, 1357–1394.
- [21] U. Pinkall and I. Sterling, *On the classification of constant mean curvature tori*, Ann. Math. **130** (1989), 407–451.
- [22] A. Pressley and G. Segal, *Loop groups*, Oxford Science Monographs, Oxford Science Publications, 1988.
- [23] H. Rosenberg, *Minimal surfaces in  $\mathbb{M}^2 \times \mathbb{R}$* , Illinois J. Math. **46** (2002), no. 4, 1177–1195.
- [24] C. Terng and K. Uhlenbeck, *Bäcklund transformations and loop group actions*, Comm. Pure and Appl. Math **LIII** (2000), 1–75.
- [25] G. Teschl, *Ordinary differential equations and dynamical systems*, Providence, RI: American Mathematical Society (AMS), 2012.
- [26] E.T. Whittaker and G.N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*. Fourth edition. Reprinted Cambridge University Press, New York 1962 vii+608

L. HAUSWIRTH, UNIVERSITÉ PARIS-EST, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, F-77454, MARNE-LA-VALLE, FRANCE  
*E-mail address:* hauswirth@univ-mlv.fr

M. KILIAN, SCHOOL OF MATHEMATICAL SCIENCES, WESTERN GATEWAY BUILDING, UNIVERSITY COLLEGE CORK, IRELAND.  
*E-mail address:* m.kilian@ucc.ie

M. U. SCHMIDT, INSTITUT FÜR MATHEMATIK, UNIVERSITÄT MANNHEIM, A5, 6, 68131 MANNHEIM, GERMANY.  
*E-mail address:* schmidt@math.uni-mannheim.de